JOURNAL OF FUNCTIONAL ANALYSIS 5, 94–124 (1970)

On Self-Adjointness in Infinite Tensor Product Spaces

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Communicated by Edward Nelson

Received January 29, 1969

I. INTRODUCTION

In Quantum Field Theory, Hamiltonians and fields are usually given as "formal" operators which can be expressed in terms of a family of canonical operators, $\{q_k\}_{k=1}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$, which are supposed to satisfy the canonical commutation relations:

$$[q_k, q_l] = 0 = [p_k, p_l], \qquad [q_k, p_l] = i\delta_{kl}. \tag{*}$$

We list the following examples:

(1) $\sum_{k=1}^{\infty} a_k q_k$

(2) $\sum_{k=1}^{\infty} b_k p_k$

(3) $\sum_{k=1}^{\infty} a_k q_k + b_k p_k$

(4) $\sum_{k=1}^{\infty} \omega_k(p_k^2 + q_k^2) + a_k q_k^4$

(5) $\sum_{k=1}^{\infty} \omega_k(p_k^2 + q_k^2) + \sum_{k,l,m,n=1}^{\infty} d_{klmn} q_k q_l q_m q_n$

($\{a_k\}$, $\{b_k\}$, $\{\omega_k\}$, $\{d_{klmn}\}$ are sequences of real numbers). The mathematical problem of making sense out of these formal operators has two parts. First, one must find a Hilbert space, X, and self-adjoint operators $\{q_k\}_{k=1}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$ on X such that (*) is true in a rigorous sense (such a structure is called a representation of the canonical commutation relations). Secondly, in such a representation one must look at the formal field and Hamiltonian operators themselves and determine whether the infinite sums converge in some sense and if so, whether the limits are self-adjoint. The first part of the problem was solved by Garding and Wightman [2] who classified all the representations of an exponential form of the relations (*). Much is known about the structure of these representations, in particular about an important subclass called the infinite tensor product representations. In this paper we show how to handle the convergence and self-adjointness questions for diagonal sums of self-adjoint operators (e.g., (1)–(4)) in the tensor product representations.

In Section 2 we discuss the self-adjointness of the sum $\sum_{k=1}^{\infty} A_k$ assuming that it converges suitably. The main result is Theorem 2.2 which gives an explicit domain of essential self-adjointness for $\sum_{k=1}^{\infty} A_k$. It is not hard to see that given any sequence of self-adjoint operators $\{A_k\}_{k=1}^{\infty}$, there always exists a sequence of numbers $\{\lambda_k\}_{k=1}^{\infty}$, and an infinite tensor-product space X such that $\sum_{k=1}^{\infty}(A_k - \lambda_k)$ is self-adjoint on X. In Section 3 we investigate under what conditions the sequence $\{\lambda_k\}_{k=1}^{\infty}$ (renormalizing constants) and the space X (a particular representation of the canonical commutation relations (CCR)) are unique.

Sections 4 and 5 are devoted to some problems in the foundations of Field Theory. In Section 4 we investigate three questions: (1) How large or small are the test function spaces for fields in different representations? (2) What are the continuity properties of the map from test functions to fields? (3) Can one find a common domain of essential self-adjointness for the fields? For example we prove that in the representations with bounded occupation numbers, all the fields $\varphi(f)$, $\pi(f)$, $f \in L^2(R^3)$ have a common domain, D, of essential self-adjointness and if $f_k \xrightarrow{L^2(R^3)} f$ then $\varphi(f_k)$ converges strongly to $\varphi(f)$ on D. This extends recent work of J. Chaiken [1].

In Section 5 we solve the field equation $(\Box + m^2)\,\varphi(x, t) = 0$ in all representations with polynomially bounded occupation numbers and investigate the regularity properties of the solution.

In Section 6 we briefly describe how to apply the self-adjointness theorems to certain kinds of nondiagonal examples like 5). We use freely the notation and results of von Neumann's theory of infinite tensor-product spaces [6] which is briefly described in the Appendix.

2. Essential Self-Adjointness

Let $\{H_k\}_{k=1}^{\infty}$ be separable Hilbert spaces, $\{A_k\}_{k=1}^{\infty}$ operators, A_k self-adjoint on H_k. $D(A_k)$ will always denote the domain of A_k,

$D^e(A_k)$ a domain on which A_k is essentially self-adjoint. Let $H = \otimes H_k$ be the infinite tensor product of the spaces H_k. H can be expressed as a direct sum (uncountable) of sub-Hilbert spaces $H(\chi)$ where $\chi = \otimes \chi_k$, $\chi_k \in H_k$, $\|\chi_k\| = 1$ and $H(\chi)$ is the closure of the finite span of $\{\psi = \otimes \psi_k\,;\ \psi_k \in H_k,\ \psi_k = \chi_k$ for $k > N$ (arbitrary)$\}$. A_k acts on each of the spaces $H(\chi)$ in a natural way (on the kth component). The problem then is: On which of the spaces $H(\chi)$, if any, can we make sense of the infinite sum $\sum_{k=1}^{\infty} A_k$?

We begin with the finite dimensional case.

LEMMA 2.1. *As an operator on* $\otimes_{k=1}^{N} H_k$, $\sum_{k=1}^{N} A_k$ *is essentially self-adjoint on* D^e, *the finite span of*

$$\left\{\psi = \bigotimes_{k=1}^{N} \psi_k\,;\ \psi_k \in D^e(A_k)\right\}.$$

Proof. We first show that $\sum_{k=1}^{N} A_k$ is essentially self-adjoint on D, the finite span of $\{\psi = \otimes_{k=1}^{N}\psi_k\,;\ \psi_k \in D(A_k)\}$. Let ψ_k be a finite vector for A_k of norm one, i.e., $E^k[-M_k, M_k]\,\psi_k = \psi_k$ for some M_k where $E^k[\mu, \nu]$ are the spectral projectors of A_k. Let $\psi = \otimes_{k=1}^{N} \psi_k$; then $\psi \in D$ and

$$\left\|\left(\sum_{k=1}^{N} A_k\right)^n \psi\right\| \leqslant \sum_{i_1+\cdots+i_n=n} \left(\frac{n!}{i_1!\cdots i_n!}\right) \prod_{k=1}^{N} \| A_k^{i_k}\psi_k \| \leqslant \left(\sum_{k=1}^{N} M_k\right)^n.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{s^n \|(\sum_{k=1}^{N} A_k)^n \psi\|}{n!} < \infty$$

for all s which shows that ψ is an analytic vector for $\sum_{k=1}^{N} A_k$. Since the set finite linear combinations of such ψ are dense in D, $\sum_{k=1}^{N} A_k$ is essentially self-adjoint on D [5].

Let $\varphi \in \otimes_{k=1}^{N} H_k$, then there exists

$$\psi = \sum_{l=1}^{L} a_l \bigotimes_{k=1}^{N} \psi_{l,k}, \qquad \psi_{l,k} \in D(A_k)$$

such that $\|(\sum_{k=1}^{N} A_k + i)\,\psi - \varphi\| < \epsilon$. Choose $\{\psi_{l,k}^n\}_{n=1}^{\infty}$ such that

$$\psi_{l,k}^n \in D^e(A_k), \qquad \psi_{l,k}^n \xrightarrow[n\to\infty]{H_k} \psi_{l,k},$$

and

$$A_k\psi^n_{l,k} \xrightarrow[n\to\infty]{H_k} A_k\psi_{l,k}\,.$$

Let $\psi^n = \sum_{l=1}^{L} a_l \otimes_{k=1}^{N} \psi^n_{l,k}$. Then $\psi^n \in D^e$ for all n and for large enough n_0,

$$\left\|\left(\sum_{k=1}^{N} A_k + i\right)\psi^{n_0} - \psi\right\| < \epsilon \quad \text{and} \quad \left\|\left(\sum_{k=1}^{N} A_k + i\right)\psi^{n_0} - \varphi\right\| < 2\epsilon.$$

Thus, $\sum_{k=1}^{N} A_k + i$ has a dense range on D^e. Since the same proof works for $\sum_{k=1}^{N} A_k - i$, $\sum_{k=1}^{N} A_k$ is essentially self-adjoint on D^e.

Suppose that there is a c_0-vector $\varphi = \otimes \varphi_k$ in $H(\chi)$ such that $\varphi_k \in D^e(A_k)$ for all k and $\lim_{N\to\infty} \sum_{k=1}^{\infty} A_k\varphi$ exists as $N \to \infty$ (we will call φ a strong convergence vector for $\{A_k\}_{k=1}^{N}$ on $H(\chi)$). Let D_φ be the finite span of the set $\{\psi = \otimes \psi_k,\ \psi_k \in D^e(A_k), \psi_k = \varphi_k$ for $k > N$ (arbitrary)$\}$. Then D_φ is a dense subset of $H(\chi)$ and $\sum_{k=1}^{N} A_k$ converges strongly on D_φ as $N \to \infty$. We denote the strong limit on D_φ by A_φ.

THEOREM 2.2. *Suppose φ is a strong convergence vector for $\{A_k\}_{k=1}^{\infty}$ on $H(\chi)$, then A_φ is essentially self-adjoint on D_φ.*

Proof. To show that A_φ is essentially self-adjoint on D_φ it is sufficient to show that any vector of the form

$$\psi = \bigotimes_{i=1}^{N_1} \psi_i \otimes \bigotimes_{i=N_1+1}^{\infty} \varphi_i$$

can be approximated by a vector in $(A_\varphi \pm i)\, D$ (since finite linear combinations of such ψ are dense in $H(\chi)$). Let η_{N_2}, $N_2 > N_1$, be a vector in $\otimes_{k=1}^{N} H_k$ made of a finite linear combination of vectors of the form $\beta_1 \otimes \cdots \otimes \beta_{N_2}$ where $\beta_i \in D^e(A_i)$, $i = 1, 2,\ldots, N_2$. Let $\eta = \eta_{N_2} \otimes \otimes_{i=N_2+1}^{\infty}\varphi_i$. Then $\eta \in D_\varphi$ and we have

$$\begin{aligned}
&\|(A_\varphi \pm i)\,\eta - \psi\,\|_{H(\chi)} \\
&\quad = \left\|\left(\sum_{n=1}^{\infty} A_n \pm i\right)\eta - \psi\right\| \leqslant \left\|\left(\sum_{n=1}^{N_2} A_n \pm i\right)\eta - \psi\right\| + \left\|\sum_{n=N_2+1}^{\infty} A_n\eta\right\| \\
&\quad = \left\|\left(\sum_{n=1}^{N_2} A_n \pm i\right)\eta_{N_2} \otimes \bigotimes_{k=N_2+1}^{\infty} \varphi_k - \psi_{N_2} \otimes \bigotimes_{k=N_2+1}^{\infty} \varphi_k\right\| \\
&\qquad + \left\|\eta_{N_2} \otimes \sum_{n=N_2+1}^{\infty} A_n\left(\bigotimes_{k=N_2+1}^{\infty} \varphi_k\right)\right\|
\end{aligned}$$

$$= \left\| \left[\left(\sum_{n=1}^{N_2} A_n \pm i \right) \eta_{N_2} - \psi_{N_2} \right] \otimes \bigotimes_{k=N_2+1}^{\infty} \varphi_k \right\|$$

$$+ \| \eta_{N_2} \|_{\otimes_{k=1}^{N_2} H_k} \cdot \left\| \sum_{n=N_2+1}^{\infty} A_n \left(\bigotimes_{k=N_2+1}^{\infty} \varphi_k \right) \right\|_{\otimes_{k=N_2+1}^{\infty} H_k}$$

$$= \left\| \left(\sum_{n=1}^{N_2} A_n \pm i \right) \eta_{N_2} - \psi_{N_2} \right\| \cdot \prod_{k=N_2+1}^{\infty} \| \varphi_k \|$$

$$+ \frac{\| \eta_{N_2} \|}{\prod_{k=1}^{N_2} \| \varphi_k \|} \cdot \left\| \sum_{n=N_2+1}^{\infty} A_n \varphi \right\|_{H(x)} .$$

Now, ϵ and ψ are given. Choose N_2 such that

$$N_2 \geqslant N_1 , \tag{1}$$

$$\frac{1}{2} \leqslant \prod_{k=N_{2+1}}^{\infty} \| \varphi_k \| \leqslant 2, \tag{2}$$

$$\left\| \sum_{k=N_{2+1}}^{\infty} A_k \varphi \right\| \leqslant \epsilon \| \varphi \|. \tag{3}$$

Then for any η of the kind we have described

$$\|(A_\infty \pm i) \eta - \psi \| \leqslant \left\| \left(\sum_{k=1}^{N_2} A_k \pm i \right) \eta_{N_2} - \psi_{N_2} \right\|_{\otimes_{k=1}^{N_2} H_k} \cdot 2 \| \varphi \|$$

$$+ 2\epsilon \| \eta_{N_2} \|_{\otimes_{k=1}^{N_2} H_k}$$

From Lemma 2.1 we know that $\sum_{k=1}^{N_2} A_k$ is essentially self-adjoint on the finite span of the set $\{\otimes_{k=1}^{N_2} \beta_k , \beta_k \in D^e(A_k)\}$ so we can choose a vector, call it $\eta_{N_2}^1$, in this set such that

$$\left\| \left(\sum_{k=1}^{N_2} A_k \pm i \right) \eta_{N_2}^1 - \psi_{N_2} \right\|_{\otimes_{k=1}^{N_2} H_k} \leqslant \epsilon \qquad \text{and} \qquad \| \eta_{N_2}^1 - \eta_{N_2}^2 \| \leqslant \epsilon$$

where $\eta_{N_2}^2$ is the resolvent of $\sum_{k=1}^{N_2} A_k$ at $\pm i$ applied to ψ_{N_2}. Let $\eta^1 = \eta_{N_2}^1 \otimes \otimes_{k=N_2+1}^{\infty} \varphi_k$.

Then $\eta^1 \in D_\varphi$ and

$$\|(A_\varphi \pm i)\,\eta^1 - \psi\|_{H(\chi)} \leqslant 2\,\|\varphi\|\,\epsilon + 2\epsilon\,(\|\eta^2_{N_2}\|_{\otimes_{k=1}^{N_2} H_k} + \epsilon).$$

Since the norm of the resolvent of $\sum_{k=1}^{N_2} A_k$ at $\pm i$ is $\leqslant 1$ we have:

$$\|\eta^2_{N_2}\|_{\otimes_{k=1}^{N_2} H_k} \leqslant \|\psi_{N_2}\|_{\otimes_{k=1}^{N_2} H_k} \leqslant 2\,\|\psi\|_{H(\chi)}\,.$$

Therefore, $\|(A_\varphi \pm i)\,\eta^1 - \psi\| \leqslant 2\,\|\varphi\|\,\epsilon + 2\epsilon^2 + 4\epsilon\,\|\psi\|$. Since ϵ is arbitrary and φ and ψ are given, we have shown that the range of $(A_\varphi \pm i)$ is dense on D_φ.

If φ and ψ are both strong convergence vectors for $\{A_k\}_{k=1}^\infty$ on $H(\chi)$ then D_φ and D_ψ are disjoint so that it is *a priori* possible that defining $\sum_{k=1}^\infty A_k$ as the closure of A_φ might give a different operator than the closure of A_ψ. But, this difficulty does not occur.

THEOREM 2.3. *Let φ and ψ be strong convergence vectors for $\{A_k\}_{k=1}^\infty$ on $H(\chi)$. Then the closure of A_φ equals the closure of A_ψ.*

Proof. For each N, $\sum_{k=1}^N A_k$ is essentially self-adjoint on D_φ and $\sum_{k=1}^N A_k$ converges to A_φ strongly on D_φ. Therefore by the Trotter–Kato theorem the group generated by $\sum_{k=1}^N A_k\,|_{D_\varphi}$ converges strongly to the group generated by the closure of A_φ. Similarly, the group generated by $\sum_{k=1}^N A_k\,|_{D_\psi}$ converges to the group generated by the closure of A_ψ. Since

$$\exp\left(it \sum_{k=1}^N A_k\,|_{D_\varphi}\right) = \prod_{k=1}^N \exp(itA_k) \otimes \bigotimes_{k=N+1}^\infty I_k = \exp\left(it \sum_{k=1}^N A_k\,|_{D_\psi}\right),$$

the groups generated by the closures of A_ψ and A_ψ are identical. Therefore the closure of A_φ equals the closure of A_ψ.

Another method for defining $\sum_{k=1}^\infty A_k$ is due to L. Streit [9]. Let $U_k(t)$ be the group generated by A_k on H_k. Then $U(t) = \otimes_{k=1}^\infty U_k(t)$ is a well-defined unitary one-parameter group (not continuous) on the entire inseparable space $H = \otimes H_k$. If $U(t)$ is reduced by any separable subspace $H(\chi)$ and if $U(t)|_{H(\chi)}$ is strongly continuous we can define $\sum_{k=1}^\infty A_k$ on $H(\chi)$ as the infinitesimal generator of $U(t)|_{H(\chi)}$. The following theorem is a straightforward generalization of a theorem of Streit. Proofs may be found in his paper [9].

THEOREM 2.4. *Let $c > 0$ and a c_0-vector $\chi = \otimes \chi_k \in H$ be given. Then for all t, $U(t)$ is reduced by $H(\chi)$ and $U(t)|_{H(\chi)}$ is a continuous* *one-parameter unitary group if and only if the following three conditions are satisfied:*

(1) $\sum_{k=1}^{\infty} |(A_k E^k[-c, c]\, \chi_k , \chi_k)| < \infty$

(2) $\sum_{k=1}^{\infty} (A_k^2 E^k[-c, c]\, \chi_k , \chi_k) < \infty$

(3) $\sum_{k=1}^{\infty} ((I - E^k[-c, c])\, \chi_k , \chi_k) < \infty$

where $E^k[\mu, \nu]$ are the spectral projectors of A_k. If the conditions (1)–(3) *are satisfied, then $U(t)|_{H(\chi)}$ is the strong limit of $\otimes_{k=1}^{N} U_k(t)$ as $N \to \infty$.*

COROLLARY 2.5. *The conditions* (1)–(3) *are satisfied if and only if there exists a c_0-vector $\varphi = \otimes \varphi_k$ in $H(\chi)$ such that $\varphi_k \in D(A_k)$ and*

$$\sum_{k=1}^{\infty} |(A_k \varphi_k \varphi_k)| < \infty, \qquad \sum_{k=1}^{\infty} \| A_k \varphi_k \|^2 < \infty.$$

We remark that the proof of Theorem 2.4 shows that if $U(t)$ is reduced by $H(\chi)$ then the restriction is continuous and $U(t)|_{H(\chi)}$ is the strong limit of $\otimes_{k=1}^{N} \exp(itA_k)$ on $H(\chi)$. The following theorem shows that the method of defining $\sum_{k=1}^{\infty} A_k$ via Theorem 2.4. is weaker than the method of defining $\sum_{k=1}^{\infty} A_k$ via Theorem 2.2 in the sense that if $U(t)$ is reduced by $H(\chi)$ then there exists a strong convergence vector for $\{A_k\}_{k=1}^{\infty}$ in $H(\chi)$ but the converse is not true. Nevertheless, Theorem 2.4 is very useful because of the explicitness of the conditions (1)–(3).

THEOREM 2.6. *$U(t)$ is reduced by $H(\chi)$ if and only if there exists a strong convergence vector, $\varphi = \otimes \varphi_k$, for $\{A_k\}_{k=1}^{\infty}$ on $H(\chi)$ satisfying*

$$\sum_{k=1}^{\infty} |(A_k \varphi_k , \varphi_k)| < \infty \tag{2.1}$$

in which case the closure of A_φ is the infinitesimal generator of $U(t)|_{H(\chi)}$. If the A_k are positive, the statement is true without the Condition (2.1).

Proof. Suppose $U(t)$ is reduced by $H(\chi)$. Corollary 2.5 gives us the existence of a c_0-vector, $\varphi = \otimes \varphi_k$, in $H(\chi)$ satisfying

$$\sum_{k=1}^{\infty} \| A_k \varphi_k \|^2 < \infty, \qquad \sum_{k=1}^{\infty} |(A_k \varphi_k , \varphi_k)| < \infty.$$

Now

$$\left\| \sum_{k=m}^{n} A_k\varphi \right\|^2 = \sum_{i\neq j=m}^{n} (A_i\varphi_i, \varphi_i)(A_j\varphi_j\varphi_j) \prod_{k\neq i,j}^{\infty} \| \varphi_k \|^2$$

$$+ \sum_{i=m}^{n} \| A_i\varphi_i \|^2 \prod_{k\neq i} \| \varphi_k \|^2. \tag{2.2}$$

Since $\prod_{k\neq i}^{\infty} \| \varphi_k \|^2$ and $\prod_{k\neq i,j}^{\infty} \| \varphi_k \|^2$ are uniformly bounded $\| \sum_{k=m}^{n} A_k\varphi \|$ can be made as small as we like for m and n large enough. Thus φ is a strong convergence vector for $\{A_k\}_{k=1}^{\infty}$ on $H(\chi)$ and it satisfies (2.1). By Lemma 2.1 and Theorem 2.2, the operators $\sum_{k=1}^{N} A_k$, $N = 1, 2, \ldots, \infty$, are all essentially self-adjoint on D_φ. Since $\sum_{k=1}^{N} A_k$ converges strongly to $\sum_{k=1}^{\infty} A_k$ on D_φ, the Trotter–Kato Theorem implies that the group generated by $\sum_{k=1}^{N} A_k$, namely $\otimes_{k=1}^{N} \exp(itA_k)$, converges strongly to the group generated by $\sum_{k=1}^{\infty} A_k$. But since $U(t)$ is reduced by $H(\chi)$, $\otimes_{k=1}^{N} \exp(itA_k)$ converges strongly to $U(t)|_{H(\chi)}$, which shows that the closure of $\sum_{k=1}^{\infty} A_k|_{D_\varphi}$ is the generator of $U(t)|_{H(\chi)}$.

Conversely if φ is a strong convergence vector for $\{A_k\}_{k=1}^{\infty}$ on $H(\chi)$ the left side of (2.2) can be made arbitrarily small for large m, n. If in addition φ satisfies (2.1) the first term on the right of (2.2) can be made small which implies that the second term will also be small for large n, m. Thus $\sum_{k=1}^{\infty} \| A_k\varphi_k \|^2 < \infty$ and the conditions of Corollary 2.5 are satisfied which proves that $U(t)$ is reduced by $H(\chi)$.

On the other hand, if φ is any c_0-vector we can choose M such that $k \geqslant M$ implies $\frac{1}{2} \leqslant \| \varphi_k \| \leqslant 2$. Thus for $m, n \geqslant M$

$$\left(\sum_{k=m}^{n} (A_k\varphi_k, \varphi_k) \right)^2 \leqslant \sum_{i\neq j=m}^{n} (A_i\varphi_i, \varphi_i)(A_j\varphi_j, \varphi_j) + \sum_{i=m}^{n} \| A_i\varphi_i \|^2 \| \varphi_i \|^2$$

$$\leqslant \frac{16}{\| \varphi \|^2} \sum_{i\neq j=m}^{n} (A_i\varphi_i, \varphi_i)(A_j\varphi_j, \varphi_j) \left(\prod_{k\neq i,j}^{\infty} \| \varphi_k \|^2 \right)$$

$$+ \frac{16}{\| \varphi \|^2} \sum_{i=m}^{n} \| A_i\varphi_i \|^2 \prod_{k\neq i}^{\infty} \| \varphi_k \|^2 = \frac{16}{\| \varphi \|^2} \left\| \sum_{k=m}^{n} A_k\varphi \right\|^2.$$

So if φ is a strong convergence vector, $\sum_{k=1}^{\infty} (A_k\varphi_k, \varphi_k) < \infty$. Therefore if the A_k are positive we need not assume (2.1).

The following simple example illustrates the conditions in Theorem 2.6. Let $A_k = [(-1)^k/k]\, I_k$, I_k the identity operator on H_k. Let χ

be any c_0-vector in H. Then every c_0-vector in $H(\chi)$ is a strong convergence vector for $\{A_k\}_{k=1}^{\infty}$ on $H(\chi)$ since

$$\left\| \sum_{k=m}^{n} [(-1)^k/k]\,\varphi \right\| = \left| \sum_{k=m}^{n} [(-1)^k/k] \right| \cdot \| \varphi \|.$$

But $U(t) = \bigotimes_{k=1}^{\infty} \exp(it[(-1)^k/k])$ is not reduced by $H(\chi)$ because $U(t)\,\chi$ is not equivalent to χ since

$$\sum_{k=1}^{\infty} | 1 - (\exp(it[(-1)^k/k]\,\chi_k , \chi_k)| = \infty.$$

Clearly $\sum_{k=1}^{\infty}(A_k\chi_k , \chi_k) < \infty$ but $\sum_{k=1}^{\infty} |(A_k\chi_k , \chi_k)| = \infty$.

Remark. A third method for defining $\sum_{k=1}^{\infty} A_k$ on $H(\chi)$ is to look for weak convergence vectors. If $\sum_{k=1}^{\infty}(A_k\varphi_k , \varphi_k) < \infty$ for some c_0-vector $\varphi = \bigotimes \varphi_k$ in $H(\chi)$ then the form $B(\psi, \eta) = \sum_{k=1}^{\infty}(A_k\psi, \eta)$ makes sense for $\psi, \eta \in D_\varphi$. If $B(\psi, \eta)$ is semi-bounded we can define $\sum_{k=1}^{\infty} A_k$ as the Friedrichs' extension of $B(\psi, \eta)$. This method will be investigated in a subsequent paper.

3. Existence and Uniqueness

In Quantum Field Theory one is often given a sequence of operators $\{A_n\}_{n=1}^{\infty}$ and the first step is to find a space $H(\chi)$ on which $\sum_{n=1}^{\infty} A_n$ is well-defined. It is therefore important to know whether such a space exists and whether it is unique. Let $\sigma(A_n)$ denote the spectrum of A_n and define $\tau_n = \inf\{| \lambda |; \lambda \in \sigma(A_n)\}$. Then we have

Proposition 3.1. *A sufficient condition that $\{A_n\}_{n=1}^{\infty}$ have at least one strong convergence vector in $H = \bigotimes_{n=1}^{\infty} H_n$ is that $\{\tau_n\}_{n=1}^{\infty} \in l_1$. It follows that for any sequence of self-adjoint operators $\{A_n\}_{n=1}^{\infty}$, there exists a sequence of numbers $\{\mu_n\}_{n=1}^{\infty}$ and a c_0-vector χ such that $\sum_{n=1}^{\infty}(A_n - \mu_n)$ is self-adjoint on $H(\chi)$.*

Proof. Let χ_n be any vector of norm one in the range of $E^n[-\tau_n - (1/n^2), \tau_n + (1/n^2)]$; $E^n[\mu, \nu]$ denotes a spectral projector of A_n. Then $\chi_n \in D(A_n)$ and $\| A_n\chi_n \| \leqslant \tau_n + (1/n^2)$. If $\chi = \bigotimes_{n=1}^{\infty} \chi_n$, then

$$\left\| \sum_{n=1}^{\infty} A_n\chi \right\| \leqslant \sum_{n=1}^{\infty} \| A_n\chi \| = \sum_{n=1}^{\infty} \| A_n\chi_n \| \leqslant \sum_{n=1}^{\infty} \tau_n + (1/n^2) < \infty$$

so χ is a strongc onvergence vector for $\{A_n\}_{n=1}^{\infty}$. For an arbitrary sequence $\{A_n\}_{n=1}^{\infty}$ we need only choose μ_n so that zero is in the spectrum of $A_n - \mu_n$ to insure that $\{A_n - \mu_n\}_{n=1}^{\infty}$ has a strong convergence vector.

Suppose the A_n are all semi-bounded (from below) and define $\kappa_n = \inf\{\lambda; \lambda \in \sigma(A_n)\}$. Then $A_n - \kappa_n$ has zero as the lowest point in its spectrum so $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ will be semi-bounded (in fact positive) on any space where $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ exists. Suppose that a sequence of real numbers $\{\eta_n\}_{n=1}^{\infty}$, satisfies $\sum_{n=1}^{\infty} \eta_n < \infty$. It is easy to see that if φ is a strong convergence vector for $\{A_n - \kappa_n\}_{n=1}^{\infty}$ on $H(\chi)$, then φ will also be a strong convergence vector for $\{A_n - (\kappa_n - \eta_n)\}_{n=1}^{\infty}$ on $H(\varphi)$ and $\sum_{n=1}^{\infty}(A_n - (\kappa_n - \eta_n))$ will be semi-bounded. Similarly, if $\sum_{n=1}^{\infty} | \eta_n | < \infty$ and $U^{\kappa}(t) = \bigotimes_{n=1}^{\infty} \exp(it(A_n - \kappa_n))$ is reduced by some $H(\chi)$, then $U^{\kappa+\eta}(t) = \bigotimes_{n=1}^{\infty} \exp(it(A_n - \kappa_n + \eta_n))$ is reduced by $H(\chi)$ and its generator is semi-bounded. The converses of these two statements show that if we require semi-boundedness the sequence $\{\kappa_n\}_{n=1}^{\infty}$ is, in a sense, unique.

THEOREM 3.2. *Let $\{\mu_n\}_{n=1}^{\infty}$ be an arbitrary sequence of real numbers, $H(\chi)$ an infinite tensor product space, then,*

(A) *If φ is a strong convergence vector for $\{A_n - \mu_n\}_{n=1}^{\infty}$ on $H(x)$ and $\sum_{n=1}^{\infty}(A_n - \mu_n)$ is semi-bounded, then φ is a strong convergence vector for $\{A_n - \kappa_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty}(\kappa_n - \mu_n) < \infty$ and*

$$\sum_{n=1}^{\infty} (A_n - \mu_n) = \sum_{n=1}^{\infty} (A_n - \kappa_n) + \sum_{n=1}^{\infty} (\kappa_n - \mu_n). \tag{3.1}$$

(B) *If $U^{\mu}(t) = \bigotimes_{n=1}^{\infty} \exp(it(A_n - \mu_n))$ is reduced by $H(\chi)$ and its generator is semi-bounded, then $U^{\kappa}(t)$ is reduced by $H(\chi)$, $\sum_{n=1}^{\infty} | \kappa_n - \mu_n | < \infty$, and* (3.1) *holds.*

Proof. We prove (A) first. If $\varphi = \bigotimes_{j=1}^{\infty} \varphi_n$ is a strong convergence vector for $\{A_n - \mu_n\}_{n=1}^{\infty}$ on $H(\chi)$ then so is $\varphi = \bigotimes_{j=1}^{\infty}(\varphi_j/\| \varphi_j \|)$, so we may assume $\| \varphi_j \| = 1$. Suppose $\sum_{j=1}^{\infty}(\mu_j - \kappa_j)$ is not conditionally convergent. Then there exists $\epsilon_0 > 0$ such that for all $M > 0$ there exist $m, n \geqslant M$ so that $| \sum_{j=m}^{n}(\mu_j - \kappa_j)| \geqslant \epsilon_0$. Since φ is a strong convergence vector for $\{A_j - \mu_j\}_{j=1}^{\infty}$, $\sum_{j=1}^{\infty}((A_j - \mu_j) \varphi_j , \varphi_j)$ is conditionally convergent (see the proof of Theorem 2.6). Choose M, such that $m, n \geqslant M$ implies $| \sum_{j=m}^{n}((A_j - \mu_j) \varphi_j , \varphi_j)| \leqslant \epsilon_0/4$. Now choose a sequence of pairs of integers $\{(m_l , n_l)\}_{l=1}^{\infty}$ such that

$$M \leqslant m_1 \leqslant n_1 < m_2 \leqslant n_2 \cdots$$

and $|\sum_{j=m_l}^{n_l}(\kappa_j - \mu_j)| \geqslant \epsilon_0$. Because

$$\left|\sum_{j=m_l}^{n_l} ((A_j - \mu_j)\varphi_j, \varphi_j)\right| = \left|\sum_{j=m_l}^{n_l} ((A_j - \kappa_j)\varphi_j, \varphi_j) + \sum_{j=m_l}^{n_l} (\kappa_j - \mu_j)\right| < \epsilon_0/4$$

we must have $\sum_{j=m_l}^{n_l}(\kappa_j - \mu_j) \leqslant -\epsilon_0$ since $A_j - \kappa_j$ is positive. Thus, the spectrum of $\sum_{j=m_l}^{n_l}(A_j - \mu_j)$ on $\bigotimes_{j=m_l}^{n_l} H_j$ contains points $\leqslant -\epsilon_0$. By Lemma 2.1, we can choose a finite linear combination, ψ^l, $\|\psi^l\| = 1$, of product vectors in

$$D(A_{m_l}) \otimes D(A_{m_{l+1}}) \cdots \otimes D(A_{n_l})$$

such that

$$\left(\sum_{j=m_l}^{n_l} (A_j - \mu_j)\psi^l, \psi^l\right) \leqslant -\epsilon_0/2.$$

Let

$$\chi^L = \bigotimes_{j=1}^{m_1-1} \varphi_j \otimes \psi^1 \otimes \bigotimes_{j=n_1+1}^{m_2-1} \varphi_j \otimes \psi^2 \otimes \cdots \otimes \psi^L \otimes \bigotimes_{j=n_L+1}^{\infty} \varphi_j .$$

Then by Theorem 2.2,

$$\chi^L \in D\left(\sum_{j=1}^{\infty} A_j - \mu_j\right), \qquad \|\chi^L\| = 1$$

and

$$\left(\sum_{j=1}^{\infty} (A_j - \mu_j)\chi^L, \chi^L\right)$$

$$= \sum_{j=1}^{m_1-1} ((A_j - \mu_j)\varphi_j, \varphi_j) + \sum_{l=1}^{L} \left(\left(\sum_{j=m_l}^{n_l} A_j - \mu_j\right)\psi^l, \psi^l\right)$$

$$+ \sum_{l=1}^{L-1} \sum_{j=n_l+1}^{m_l-1} ((A_j - \mu_j)\varphi_j, \varphi_j) + \sum_{j=n_L+1}^{\infty} ((A_j - \mu_j)\varphi_j, \varphi_j)$$

$$\leqslant \frac{-L\epsilon_0}{4} + \sum_{j=1}^{\infty} ((A_j - \mu_j)\varphi_j, \varphi_j).$$

Since the second term on the right converges and in the first term L is arbitrary, we have shown that the spectrum of $\sum_{j=1}^{\infty} A_j - \mu_j$ on $H(\chi)$ is unbounded below which contradicts the hypothesis of (A). Thus $\sum_{j=1}^{\infty}(\kappa_j - \mu_j)$ converges conditionally. The rest of (A) follows trivially.

To prove (B) we observe that if $U^\mu(t)$ is reduced by $H(\chi)$ then by

Theorem 2.6, $\{A_j - \mu_j\}_{j=1}^{\infty}$ has a strong convergence vector, φ, in $H(\chi)$ satisfying

$$\sum_{j=1}^{\infty} |((A_j - \mu_j)\varphi_j, \varphi_j)| < \infty.$$

Suppose $\sum_{j=1}^{\infty} | \kappa_j - \mu_j | = \infty$ and let $P = \{j; \kappa_j - \mu_j \geqslant 0\}$. Then $\sum_{j \in P}(\kappa_j - \mu_j) < \infty$ since otherwise

$$\sum_{j \in P} |((A_j - \mu_j)\varphi_j, \varphi_j)| = \sum_{j \in P} ((A_j - \kappa_j)\varphi_j, \varphi_j) + (\kappa_j - \mu_j) = \infty.$$

Therefore $\sum_{j \in \tilde{P}}(\kappa_j - \mu_j) = -\infty$. For $j \in P$, define $\psi_j = \varphi_j$, and for $j \in \tilde{P}$ choose ψ_j so that $((A_j - \mu_j)\psi_j, \psi_j) \leqslant \frac{1}{2}(\kappa_j - \mu_j)$. Let $\chi^L = \otimes_{j=1}^{L} \psi_j \otimes \otimes_{j=L+1}^{\infty} \varphi_j$, then $\chi^L \in D(\sum_{j=1}^{\infty}(A_j - \mu_j))$ and $(\sum_{j=1}^{\infty}(A_j - \mu_j)\chi^L, \chi^L)$ can be made as small as we like. Since by Theorem 2.6 $\sum_{j=1}^{\infty} A_j - \mu_j$ is the infinitesimal generator of $U^{\mu}(t)|_{H(\chi)}$, this contradicts the semi-boundedness hypothesis in (B). Thus $\sum_{j=1}^{\infty} | \kappa_j - \mu_j | < \infty$ and the rest of the statements in (B) follow easily.

We now restrict our attention to the sequence $\{A_n - \kappa_n\}_{n=1}^{\infty}$. Since $A_n - \kappa_n$ is positive, $\{A_n - \kappa_n\}_{n=1}^{\infty}$ will have a strong convergence vector in a space $H(\chi)$ if and only if $U^{\kappa}(t)$ is reduced by $H(\chi)$. So, when either condition is satisfied we will say merely that $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ is self-adjoint on $H(\chi)$. Now, suppose χ and χ' are weakly equivalent c_0-vectors, $\chi \approx \chi'$, then there is a sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$, $| z_n | = 1$, such that $\otimes_{n=1}^{\infty} z_n\chi \sim \otimes_{n=1}^{\infty} \chi_n'$ (strong equivalence). If $U^{\kappa}(t)$ is reduced by $H(\chi)$, then $U^{\kappa}(t)\chi \sim \chi$ and $\sum_{n=1}^{\infty} | 1 - (U_n(t)\chi_n, \chi_n)| < \infty$ where $U_n(t) = \exp(it(A_n - \kappa_n))$. But, then $\sum_{n=1}^{\infty} | 1 - (U_n(t) z_n\chi_n, z_n\chi_n)| < \infty$ which proves that $U^{\kappa}(t) \otimes_{n=1}^{\infty} z_n\chi_n \sim \otimes_{n=1}^{\infty} z_n\chi_n$. Since $\otimes_{n=1}^{\infty} z_n\chi_n$ is a c_0-vector in $H(\chi')$ this implies (see the proof of Theorem 2.4) that $U^{\kappa}(t)$ is reduced by $H(\chi')$. In other words, if $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ is self-adjoint on a space $H(\chi)$, then $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ is also self-adjoint on all spaces, $H(\chi')$, generated by c_0-vectors which are weakly equivalent to χ. We will call such spaces weakly equivalent to $H(\chi)$. The question then arises under what conditions is $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ self-adjoint on only one weak equivalence class of spaces. In the case of interest to Quantum Field Theory each weak equivalence class corresponds to a different representation of the canonical commutation relation [*4*], so we are asking under what conditions is $\sum_{n=1}^{\infty} A_n - \kappa_n$ well-defined and self-adjoint in one and only one tensor product representation of the canonical commutation relations.

THEOREM 3.3. *Let* $\kappa_n = \inf \sigma(A_n)$, *then* $\sum_{n=1}^{\infty} A_n - \kappa_n$ *is self-adjoint on a unique weak equivalence class of spaces,* $H(\chi)$, *if and only if both of the following conditions hold:*

(a) *For all but a finite number of* n, κ_n *is an isolated point spectrum of* A_n *of multiplicity one.*

(b) *Let* $\tau_n = \inf\{\lambda; \lambda \in \sigma(A_n), \lambda \neq \kappa_n\}$, *then*

$$\{\mu_n(\tau_n - \kappa_n)^{1/2}\}_{n=1}^{\infty} \in l_2 => \{\mu_n\}_{n=1}^{\infty} \in l_2$$

for any sequence of numbers $\{\mu_n\}_{n=1}^{\infty}$.

Proof. We will prove the necessity of (a) and (b) first. Choose χ_n^1 and χ_n^2 in the range of $E^n[\kappa_n, \kappa_n + (1/n^2)]$ on H_n, $\|\chi_n^1\| = \|\chi_n^2\| = 1$, and if possible orthogonal. Then it is easy to see that both $\chi^1 = \otimes_{n=1}^{\infty} \chi_n^1$ and $\chi^2 = \otimes_{n=1}^{\infty} \chi_n^2$ will be strong convergence vectors for $\{A_n - \kappa_n\}_{n=1}^{\infty}$. However, if for more than a finite number of n, κ_n is not isolated or has multiplicity greater than one, then $\chi^1 \not\approx \chi^2$ which shows that (a) is necessary.

To simplify the proof we now assume that all κ_n are isolated point spectrum of multiplicity one (only the behavior of the spectrum of A_n for large n matters). Suppose a sequence $\{\mu_n\}_{n=1}^{\infty}$ existed such that $\{\mu_n\}_{n=1}^{\infty} \notin l_2$ but $\{\mu_n(\tau_n - \mu_n)^{1/2}\}_{n=1}^{\infty} \in l_2$. We may assume $|\mu_n| < 1$. Let $\lambda_n = (1 - |\mu_n|^2)^{1/2}$, $\chi_n^1 = E^n[\kappa_n] H_n$, $\|\chi^1\| = 1$, and

$$\chi_n^2 \in E^n[\tau_n, \tau_n + (1/n^2)] H_n, \qquad \|\chi_n^2\| = 1,$$

and set $X_n^3 = \lambda_n\chi_n^1 + \mu_n\chi_n^2$. Then

$$\sum_{n=1}^{\infty} ((A_n - \kappa_n)\chi_n^3, \chi_n^3) = \sum_{n=1}^{\infty} |\mu_n|^2 \int_{\tau_n \leqslant \sigma \leqslant \tau_n + (1/n^2)} (\sigma - \kappa_n)\, d_\sigma(E_\sigma^n \chi_n^2, \chi_n^2)$$

$$\leqslant \sum_{n=1}^{\infty} |\mu_n|^2 (\tau_n - \kappa_n) + \sum_{n=1}^{\infty} |\mu_n|^2 (1/n^2) < \infty$$

Since $\sum_{n=1}^{\infty}((A_n - \kappa_n)\chi_n^3, \chi_n^3) < \infty$, there exists M such that $n \geqslant M$ implies $\| E^n[\kappa_n, \kappa_n + 1]\chi_n^3\| \geqslant \frac{1}{2}$. For $n < M$ set $\chi_n^4 = \chi_n^3$ and for $n \geqslant M$ set $\chi^4 = E^n[\kappa_n, \kappa_n + 1]\chi_n^3$, and let $\chi^4 = \otimes_{n=1}^{\infty} \chi_n^4$. Then,

$$\sum_{n=1}^{\infty} |1 - (\chi_n^4, \chi_n^3)| \leqslant \sum_{n=1}^{\infty} \int_{\sigma > \kappa_n + 1} d_\sigma(E_\sigma^n \chi_n^3, \chi_n^3)$$

$$\leqslant \sum_{n=1}^{\infty} \int_{\sigma \geqslant \kappa_n} (\sigma - \kappa_n)\, d_\sigma(E_\sigma^n \chi_n^3, \chi_n^3)$$

$$= \sum_{n=1}^{\infty} ((A_n - \kappa_n)\chi_n^3, \chi_n^3) < \infty$$

so $\chi^4 \sim \chi^3$. But for $n \geqslant M$, $((A_n - \kappa_n)\chi_n^4, \chi_n^4) \leqslant ((A_n - \kappa_n)\chi_n^3, \chi_n^3)$ and $((A_n - \kappa_n)^2 \chi_n^4, \chi_n^4) \leqslant ((A_n - \kappa_n)\chi_n^4, \chi_n^4)$. Thus

$$\sum_{n=1}^{\infty} ((A_n - \kappa_n)\chi_n^4, \chi_n^4) < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} ((A_n - \kappa_n)^2 \chi_n^4, \chi_n^4) < \infty.$$

Corollary 2.5 shows that $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ is self-adjoint on $H(\chi^4) = H(\chi^3)$. $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ is trivially self-adjoint on $H(\chi^1)$. Now, $(\chi^1, \chi^3) = \prod_{n=1}^{\infty}(\chi_n^1, \lambda_n\chi_n^1 + \mu_n\chi_n^2) = \prod_{n=1}^{\infty} | \lambda_n |$ which converges absolutely if and only if $\sum_{n=1}^{\infty} | 1 - | \lambda_n | | < \infty$. But we assumed $\{\mu_n\}_{n=1}^{\infty} \notin l_2$ which implies that

$$\sum_{n=1}^{\infty} | 1 - | \lambda_n || = \sum_{n=1}^{\infty} | 1 - (1 - | \mu_n |^2)^{1/2} | = \infty.$$

Therefore $H(\chi^1) \not\approx H(\chi^3)$ and uniqueness does not hold. Thus (b) is necessary.

Now, suppose (a) and (b) are true (again for simplicity we assume the condition in (a) holds for all n). Suppose $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ is self-adjoint on $H(\chi^5)$. Then there exists a c_0-vector $\chi^6 = \otimes_{n=1}^{\infty} \chi_n^6 \in H(\chi^5)$ such that $\sum_{n=1}^{\infty}((A_n - \kappa_n)\chi_n^6, \chi_n^6) < \infty$. Now, χ_n^6 can be written $\chi_n^6 = \lambda_n\chi_n^1 + \mu_n\chi_n^7$ where $| \lambda_n |^2 + | \mu_n |^2 = 1$, $(A_n - \kappa_n)\chi_n^1 = 0$,

$$\chi_n^7 \in E^n[\tau_n, \infty), \qquad \| \chi_n^7 \| = \| \chi_n^1 \| = 1.$$

But,

$$((A_n - \kappa_n)\chi_n^6, \chi_n^6) = | \mu_n |^2 \int_{\sigma \geqslant \tau_n} (\sigma - \kappa_n) d_\sigma(E_\sigma^n \chi_n^7, \chi_n^7) \geqslant | \mu_n |^2 (\tau_n - \kappa_n).$$

Thus $\sum_{n=1}^{\infty} | \mu_n |^2(\tau_n - \kappa_n) < \infty$ so by (b) we must have $\{\mu_n\}_{n=1}^{\infty} \in l_2$. However, this implies $\sum_{n=1}^{\infty} | 1 - (1 - | \mu_n |^2)^{1/2} | < \infty$ so

$$\prod_{n=1}^{\infty} |(\chi_n^1, \chi_n^6)| = \prod_{n=1}^{\infty} (1 - | \mu_n |^2)^{1/2}$$

converges which implies $H(\chi^1) \approx H(\chi^6)$. Thus, $\sum_{n=1}^{\infty}(A_n - \kappa_n)$ is self-adjoint on one and only one weak equivalence class of spaces.

4. Large and Small Test Function Spaces for Quantum Fields

Let $H = \otimes_{n=1}^{\infty} L^2(R_n)$, where R_n denotes a copy of the real numbers. If q_n denotes multiplication by x_n on $L^2(R_n)$ and p_n the operator $(1/i)\, d/dx_n$, then the set of operators $\{q_n, p_n\}_{n=1}^{\infty}$ is a representation of the CCR on each of the separable subspaces $H(\chi) \subset H$.

The representations on two spaces $H(\chi)$ and $H(\chi')$ are equivalent if and only if χ is weakly equivalent to X' [4]; by equivalence we mean that there exists a unitary operator $U : H(\chi) \to H(\chi')$ such that $q_k|_{H(\chi')} = Uq_k|_{H(\chi)} U^{-1}$ and $p_k|_{H(\chi')} = Up_k|_{H(\chi)} U^{-1}$. Now, if $a = \{a_n\}_{n=1}^{\infty}$ is any sequence of real numbers we can choose a representation such that $\varphi(a) = \sum_{n=1}^{\infty} a_n q_n$ is self-adjoint (see Proposition 3.1), but in Field Theory the question arises from a different point of view. In a given representation we wish to know for what sequences, $a = \{a_n\}_{n=1}^{\infty}$, $\varphi(a)$ is self-adjoint and what are the properties of the map $a \to \varphi(a)$ (the sequences $\{a_n\}_{n=1}^{\infty}$ correspond to the test functions with which the field is smeared in the usual formulation). Similar questions arise for the conjugate momentum, $\pi(b) = \sum_{n=1}^{\infty} b_n p_n$. In this section we will use as a criterion for the self-adjointness of $\sum_{n=1}^{\infty} A_n$ the more restrictive conditions of Theorem 2.4 (instead of Theorem 2.2). This is convenient because we get nicer test sequence spaces: For example, it is not hard to construct a representation where $\{a_n q_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 2.4 if and only if $\{a_n\}_{n=1}^{\infty} \in l_1$ but $\{a_n q_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 2.2 if and only if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Definition. We denote by $\tau(\varphi, \chi)$, $\tau(\pi, \chi)$, $\tau(A, \chi)$, respectively sets of sequences such that $\varphi(a)$, $\pi(a)$, or in general $\sum_{n=1}^{\infty} a_n A_n$ is self-adjoint on $H(\chi)$.

From the conditions in Theorem 2.4, it is easy to see that $\tau(A, \chi)$ will always be a linear space which contains the finite sequences. Let $S = \{a = \{a_n\}_{n=1}^{\infty} ;\ \| a \|_r^2 = \sum_{n=1}^{\infty} | a_n |^2 (1 + n)^r < \infty$ for all positive integers $r\}$. With the semi-norms, $\| \cdot \|_r$, S is locally convex, we denote its strong dual by S'. It is not hard to see that S' is just the set of sequences $\{a_n\}_{n=1}^{\infty}$ with $| a_n | \leqslant (1 + n)^r$ for some positive integer r.

Theorem 4.1. *Let* $\{A_n\}_{n=1}^{\infty}$, A_n *self-adjoint on* H_n, *and* $H(\chi) \subset \bigotimes_{n=1}^{\infty} H_n$ *be given. Then*

(A) $\tau(\chi, A) \supseteq S$ *if and only if there exists a* c_0*-vector*

$$\varphi = \bigotimes_{n=1}^{\infty} \varphi_n \in H(\chi) \quad \text{such that} \quad \varphi_n \in D(A_n) \quad \text{and} \quad \{\| A_n \varphi_n \|\}_{n=1}^{\infty} \in S'.$$

In this case there exists a domain, D, on which each of the family of operators $\{\sum_{n=1}^{\infty} a_n A_n ;\ \{a_n\}_{n=1}^{\infty} \in S\}$ *is essentially self-adjoint. Furthermore, if* $a^k \in S$ *and* $a^k \to a \in S$ *then* $\sum_{n=1}^{\infty} a_n{}^k A_n \to \sum_{n=1}^{\infty} a_n A_n$ *strongly on* D.

(B) *$\tau(A, \chi) \supseteq S'$ if and only if there exists a c_0-vector*

$$\varphi = \bigotimes_{n=1}^{\infty} \varphi_n \in H(\chi)$$

such that $\varphi_n \in D(A_n)$ and $\{\| A_n\varphi_n \|\}_{n=1}^{\infty} \in S$. In this case, there exists a domain D on which all the operators of the family

$$\left\{\sum_{n=1}^{\infty} a_n A_n \; ; \{a_n\}_{n=1}^{\infty} \in S'\right\}$$

are essentially self-adjoint. Furthermore if a^ν is a net in S' which converges weakly to $a \in S'$ then $\sum_{n=1}^{\infty} a_n^\nu A_n$ converges strongly to $\sum_{n=1}^{\infty} a_n A_n$ on D.

Proof. We prove (A) first. Suppose $\varphi \in H(\chi)$, $\varphi_n \in D(A_n)$ and $\{\| A_n\varphi_n \|\}_{n=1}^{\infty} \in S'$. Then for some positive integer r and constant d,

$$d(n+1)^r \geqslant \int_{-\infty}^{\infty} \lambda^2 \, d_\lambda(E_\lambda{}^n\varphi_n , \varphi_n) \geqslant (n+1)^{2(r+l)} \int_{|\lambda| \geqslant (1+n)^{r+l}} d_\lambda(E_\lambda{}^n\varphi_n , \varphi_n).$$

Choose $l \geqslant 2$ and define $\psi_n = E^n[-(1+n)^{r+l}, (1+n)^{r+l}] \, \varphi_n$. Then

$$\sum_{n=1}^{\infty} | 1 - (\varphi_n , \psi_n)| \geqslant \sum_{n=1}^{\infty} \frac{d}{(1+n)^{r+l}} < \infty \qquad \text{so} \qquad \psi = \otimes\psi_n \in H(\chi),$$

$\psi_n \in D(A_n{}^j)$ for all j, and since $\| A_n\psi_n \| \leqslant \| A_n\varphi_n \|$ we have $\{\| A_n\psi_n \|\}_{n=1}^{\infty} \in S'$. Thus, if $\{a_n\}_{n=1}^{\infty} \in S$,

$$\sum_{n=1}^{\infty} |(a_n A_n\psi_n , \psi_n)| \leqslant \sum_{n=1}^{\infty} | a_n | \, \| A_n\psi_n \| \, \| \psi_n \| < \infty$$

(since $\| \psi_n \|$ is bounded) and $\sum_{n=1}^{\infty}(A_n{}^2\psi_n , \psi_n) < \infty$. Therefore by Corollary 2.5, $\sum_{n=1}^{\infty} a_n A_n$ is self-adjoint on $H(\chi)$ which implies $\tau(\chi, A) \supseteq S$.

Suppose $\tau(\chi, A) \supseteq S$ and that for each positive integer r,

$$\sum_{n=1}^{\infty} \int_{|\lambda| \geqslant n^r} d_\lambda(E_\lambda{}^n\chi_n , \chi_n) = \infty.$$

There exists an integer k_1 such that $\sum_{n=1}^{k_1} \int_{|\lambda| \geqslant n^1} d_\lambda(E_\lambda{}^n\chi_n , \chi_n) \geqslant \frac{1}{2}$. Having chosen $k_{l+1} > k_l$ so that $\sum_{n=k_l+1}^{k_{l+1}} \int_{|\lambda| \geqslant n^{l+1}} d_\lambda(E_\lambda{}^n\chi_n , \chi_n) \geqslant \frac{1}{2}$, define a sequence $a_n = 1/n^{l+1}$, for $k_l < n \leqslant k_{l+1}$; then $\{a_n\}_{n=1}^{\infty} \in S$ and

$$\sum_{n=1}^{\infty} \left(\left(I - E^n\left[-\frac{1}{|a_n|}, \frac{1}{|a_n|}\right]\right) \chi_n, \chi_n\right)$$

$$= \sum_{l=0}^{\infty} \sum_{n=k_l+1}^{k_{l+1}} \int_{|\lambda| \geqslant n^{l+1}} d_\lambda(E_\lambda^n \chi_n, \chi_n) = \infty.$$

But, this violates Condition (3) of Theorem 2.4 which must hold since $\sum_{n=1}^{\infty} a_n A_n$ is self-adjoint on $H(\chi)$. Thus, there exists an r such that

$$\sum_{n=1}^{\infty} \int_{|\lambda| \geqslant n^r} d_\lambda(E_\lambda^n \chi_n, \chi_n) < \infty. \tag{4.1}$$

Let M be an integer such that $n \geqslant M$ implies

$$\int_{|\lambda| \geqslant n^r} d_\lambda(E_\lambda^n \chi_n, \chi_n) \leqslant \tfrac{1}{2}$$

and define φ_n to be any vector in $D(A_n)$ if $n < M$ and $\varphi_n = E^n[-n^r, n^r]\, \chi_n$ if $n \geqslant M$. Then, (4.1) implies that

$$\varphi = \bigotimes_{n=1}^{\infty} \varphi_n \in H(\chi)$$

and since $\| A_n \varphi_n \| \leqslant n^r$ we have $\{\| A_n \varphi_n \|\}_{n=1}^{\infty} \in S'$. This completes the proof of the if-and-only-if statement of (A).

If the condition of (A) holds, then the vector $\psi = \otimes_{n=1}^{\infty} \psi_n$ (constructed above) is a strong convergence vector for all of the operators $\{\sum_{n=1}^{\infty} a_n A_n, \{a_n\}_{n=1}^{\infty} \in S\}$ since $\sum_{n=1}^{\infty} |((a_n A)^j \psi_n, \psi_n)| < \infty$ $j = 1, 2$. By Theorem 2.2 they are all essentially self-adjoint on D_ψ, the domain generated by ψ. Suppose $a^k \xrightarrow{S} a$ as $k \to \infty$ and let $\eta \in D_\psi$. Then η has the form $\eta = \eta_M \otimes_{n=M+1}^{\infty} \psi_n$ where η_M is a finite sum of vectors in $D(A_1) \otimes \cdots \otimes D(A_M)$. Let $b_n^k = a_n^k$ if $n \leqslant m$ and $b_n^k = 0$ if $n > M$ and let $c_n^k = a_n^k - b_n^k$. Then $a^k = b^k + c^k$. Similarly we write $a = b + c$. Since $b_n^k \to b_n$ as $k \to \infty$, $b^k \xrightarrow{S} b$, which implies $c^k \to c$. Thus

$$\left\|\left(\sum_{n=1}^{\infty} a_n^k A_n \eta - \sum_{n=1}^{\infty} a_n A_n \eta\right\| \leqslant \left(\sum_{n=1}^{M} | b_n^k - b_n | \| A_n \eta_M \|\right) \prod_{l=M+1}^{\infty} \| \psi_l \|$$

$$+ \sum_{n=M+1}^{\infty} | c_n^k - c_n | \| A_n \psi_n \| \cdot \| \eta_M \| \cdot \| \psi_n \|^{-1} \prod_{l=M+1}^{\infty} \| \psi_l \|.$$

Since $\| \psi_n \|^{-1}$ is bounded and $\{\| A_n\psi_n \|\} \in S'$, the right-hand side converges to zero as $k \to \infty$. Thus, we have proven (A).

The proof of (B) is similar to the proof of (A); we sketch only the construction of $\varphi = \prod \otimes_{n=1}^{\infty} \varphi_n$ in the proof of necessity. Suppose $\tau(A, \chi) \supseteq S'$, and suppose that there exists a positive integer r such that

$$\sum_{n=1}^{\infty} \int_{|\lambda|>1/n^r} d_\lambda(E_\lambda{}^n\chi_n, \chi_n) = \infty.$$

Let $a_n = n^r$; then $\{a_n\}_{n=1}^{\infty} \in S'$ but the sequence $\{a_nA_n\}_{n=1}^{\infty}$ does not satisfy Condition (3) of Theorem 2.4 for $c = 1$ which contradicts the assumption that $\tau(A, \chi) \supseteq S'$. Thus

$$\sum_{n=1}^{\infty} \int_{|\lambda|>1/n^r} d_\lambda(E_\lambda{}^n\chi_n, \chi_n) < \infty$$

for all positive integers r. Choose an integer k_1 such that

$$\sum_{n=k_1}^{\infty} \int_{|\lambda|>1/n} d_\lambda(E_\lambda{}^n\chi_n, \chi_n) \leqslant \tfrac{1}{2}.$$

Having chosen k_l choose $k_{l+1} > k_l$ such that

$$\sum_{n=k_{l+1}}^{\infty} \int_{|\lambda|>1/n^{l+1}} d_\lambda(E_\lambda{}^n\chi_n, \chi_n) > 1/2^{l+1}.$$

For $n < k_1$, let φ_n be any vector in $D(A_n)$; for $k_l \leqslant n \leqslant k_{l+1} - 1$, define $\varphi_n = E^n[-1/n^l, 1/n^l]\, \chi_n$. Then

$$\sum_{n=k_1}^{\infty} |\, 1 - (\varphi_n, \chi_n)| = \sum_{l=1}^{\infty} \sum_{n=k_l}^{k_{l+1}-1} \int_{|\lambda|>1/n^l} d_\lambda(E_\lambda{}^n\chi_n, \chi_n) \leqslant \sum_{l=1}^{\infty} 1/2^l < \infty$$

which implies that $\varphi = \otimes_{n=1}^{\infty} \varphi_n \in H(\chi)$. Clearly $\varphi_n \in D(A_n)$ and since $\| A_n\varphi_n \| \leqslant 1/n^l$ for $n \geqslant k_l$ we have $\{\| A_n\varphi_n \|\}_{n=1}^{\infty} \in S$.

We note that by the Trotter–Kato Theorem, the continuity of the map $\{a_n\}_{n=1}^{\infty} \to \sum_{n=1}^{\infty} a_nA_n$ proven here is stronger than is usually proven in Quantum Field Theory, namely that convergence in the test function space implies convergence of the exponentiated fields (see [1] or [3]). Since Schwartz space, $\mathscr{S}(R^3)$, is isomorphic to S [8], Theorem 4.1 shows how to construct representations where the field, $\varphi(a) = \sum_{n=1}^{\infty} a_nq_n$, or its conjugate momentum, $\pi(b) = \sum_{n=1}^{\infty} b_np_n$, can be smeared with very singular test functions (in fact, all tempered distributions). However, the theorem applies separately to φ and π, that is, if the test function space for φ is large, the test function space for π will be small. This statement is made precise by the following theorem.

THEOREM 4.2. *In any infinite tensor product representation of the canonical commutation relations:*

(a) $\tau(\varphi, \chi) \supseteq S' \Rightarrow \tau(\pi, \chi) \subseteq S$,

(b) $\tau(\varphi, \chi) \supseteq S \Rightarrow \tau(\pi, \chi) \subseteq S'$,

(c) $\tau(\pi, \chi) \supseteq S' \Rightarrow \tau(\varphi, \chi) \subseteq S$,

(d) $\tau(\pi, \chi) \supseteq S \Rightarrow \tau(\varphi, \chi) \subseteq S'$.

Proof. We suppose $\tau(\varphi, \chi) \supseteq S'$ then by Theorem 4.1 there is a c_0-vector, $\varphi = \otimes_{n=1}^{\infty} \varphi_n \in H(\chi)$ such that $\{\| q_n\varphi_n \|\}_{n=1}^{\infty} \in S$. Suppose $\{b_n\}_{n=1}^{\infty} \notin S$. We will show that $\otimes_{n=1}^{\infty} \exp(itb_n p_n)\, \varphi \not\sim \varphi$ which shows that $\otimes_{n=1}^{\infty} \exp(itb_n p_n)$ is not reduced by $H(\chi)$; i.e. that $\{b_n\}_{n=1}^{\infty} \notin \tau(\pi, \chi)$. Since $\{b_n\}_{n=1}^{\infty} \notin S$, there is a constant a and a positive integer l such that $| b_n | \geqslant a/n^l$ for $n \in P$, an infinite set of positive integers. Since $\{\| q_n\varphi_n \|\}_{n=1}^{\infty} \in S$, we can choose a constant so that for all n

$$\frac{d}{n^{3(l+1)}} \geqslant \| q_n\varphi_n \|^2 \geqslant \int_{|x| \geqslant 1/n^{l+1}} x^2 \,| \varphi_n(x)|^2 \, dx \geqslant \frac{1}{n^{2(l+1)}} \int_{|x| \geqslant 1/n^{l+1}} | \varphi_n(x)|^2 \, dx$$

or

$$\int_{|x| \geqslant 1/n^{l+1}} | \varphi_n(x)|^2 \, dx \leqslant \frac{d}{n^{l+1}} \,.$$

Thus

$$\begin{aligned} |(\exp(itb_n p_n)\, \varphi_n \,, \varphi_n)| &\leqslant \int | \varphi_n(x + b_n t)\, \varphi(x)| \, dx \\ &\leqslant \left(\int_{|x| \leqslant 1/n^{l+1}} | \varphi_n(x + b_n t)|^2 \, dx\right)^{1/2} \cdot \left(\int_{|x| \leqslant 1/n^{l+1}} | \varphi_n(x)|^2 \, dx\right)^{1/2} \\ &\quad + \left(\int_{|x| \geqslant 1/n^{l+1}} | \varphi_n(x + b_n t) \,|^2 \, dx\right)^{1/2} \cdot \left(\int_{|x| \geqslant 1/n^{l+1}} | \varphi(x)|^2 \, dx\right)^{1/2} \\ &\leqslant \| \varphi_n \| \left(\int_{|x| \leqslant 1/n^{l+1}} | \varphi_n(x + b_n t)|^2 \, dx\right)^{1/2} + \left(\frac{d}{n^{l+1}}\right)^{1/2} \| \varphi_n \|. \end{aligned}$$

Now fix $t > 0$ and choose N such that $n \geqslant N$ implies that $ta/n^l \geqslant 2/n^{l+1}$. Then for $n \in P$, $n \geqslant N$, and $b_n > 0$ we have

$$\begin{aligned}\int_{|x| \leqslant 1/n^{l+1}} &| \varphi_n(x + b_n t)|^2 \, dx \\ &= \int_{b_n t-(1/n^{l+1})}^{b_n t+(1/n^{l+1})} | \varphi_n(y)|^2 \, dy \leqslant \int_{at/n^l-(1/n^{l+1})}^{\infty} | \varphi_n(y)|^2 \, dy \\ &\leqslant \int_{1/n^{l+1}}^{\infty} | \varphi_n(y)|^2 \, dy \leqslant \frac{d}{n^{l+1}} \| \varphi_n \|^2.\end{aligned}$$

Similarly

$$\int_{|x| \leqslant 1/n^{l+1}} | \varphi_n(x + b_n t)|^2 \, dx \leqslant \frac{d}{n^{l+1}} \| \varphi_n \|^2 \qquad \text{if} \quad b_n < 0.$$

Thus for $n \in P$ and $n \geqslant N$,

$$|(\exp(itb_n p_n)\, \varphi_n \,, \varphi_n)| \leqslant 2(1/n^{l+1})^{1/2} \| \varphi_n \|.$$

Since $\| \varphi_n \| \to 1$ as $n \to \infty$, there are infinitely many n for which $|(\exp(itb_n p_n)\, \varphi_n \,, \varphi_n)| \leqslant \frac{1}{2}$ which implies that $\otimes \exp(itb_n p_n)\, \varphi \not\to \varphi$. Thus $\{b_n\}_{n=1}^{\infty} \notin \tau(\pi, \chi)$, which implies $\tau(\pi, \chi) \subseteq S$.

To prove (b) suppose that $\tau(\varphi, \chi) \supseteq S$. Then Theorem 4.1 shows that there is a c_0-vector $\varphi = \otimes \varphi_n \in H(\chi)$ such that $\{\| q_n \varphi_n \|^2\}_{n=1}^{\infty} \in S'$. Further, it follows from the proof of Theorem 4.1 that there is a positive integer r such that $\int_{|x| \geqslant n^r} | \varphi_n(x)|^2 \, dx \leqslant d/n^r$ for some d and all n. Now let $\{b_n\}$ be a sequence of real numbers, $\{b_n\}_{n=1}^{\infty} \notin S'$. Then for an infinite number of n, $| b_n | \geqslant n^{r+1}$. Using these inequalities it is easy to show as in the proof of (a) that

$$|(\exp(itb_n p_n)\, \varphi_n \,, \varphi_n)| = \left| \int_{-\infty}^{\infty} \varphi_n(x + tb_n)\, \overline{\varphi_n(x)} \, dx \right|$$

will be small for large n which implies $\otimes \exp(itb_n p_n)\, \varphi \not\to \varphi$, and $\{b_n\} \notin \tau(\pi, \chi)$. Thus $\tau(\pi, \chi) \subseteq S'$.

Relations (c) and (d) follow from the symmetry of q_n and p_n under the Fourier transform.

In certain representations the test function space for the field and the space for the conjugate momentum (or any linear combination) can both be made reasonably large. Let $\{h_k(x); k = 0, 1, 2,...\}$ denote the normalized Hermite functions. A representation of the CCR on a space $H(\chi)$ is said to have bounded occupation numbers, if $\chi = \otimes_{k=1}^{\infty} h_{n_k}(x)$, where $\{n_k\}_{k=0}^{\infty}$ is any sequence of nonnegative integers satisfying $\sup_k \{n_k\} < \infty$. The Fock representation is just the case where $n_k = 0$ for all k. The following theorem shows that as far as test function spaces for the field are concerned, representations with bounded occupation numbers have the same nice properties as the Fock representation. The continuity statement extends a recent result of J. Chaiken [1].

THEOREM 4.3. *Let* $\chi = \otimes_{k=1}^{\infty} h_{n_k}(x)$ *and* $\sup_k n_k < \infty$. *If* $(\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty}) \in l_2 \times l_2$, *then* $\sum_{k=1}^{\infty} a_k q_k + b_k p_k$ *is self-adjoint on* $H(\chi)$. *Further, there exists a domain* $D_0 \subset H(\chi)$ *on which all the operators* $\{\sum_{k=1}^{\infty} a_k q_k + b_k p_k ; (\{a_k\}, \{b_k\}) \in l_2 \times l_2\}$ *are essentially self-adjoint. If* $(a^n, b^n) \xrightarrow{l_2 \times l_2} (a, b)$, *then* $\sum_{k=1}^{\infty} a_k^n q_k + b_k^n p_k \to \sum_{k=1}^{\infty} a_k q_k + b_k p_k$ *strongly on* D_0.

Proof. Suppose $(\{a_k\}, \{b_k\}) \in l_2 \times l_2$. Since the Hermite functions are either even or odd and invariant under the Fourier transform $((a_k q_k + b_k p_k) h_{n_k}, h_{n_k}) = 0$. Further, since h_{n_k} is one of a finite collection of Hermite functions which appear in the vector $\chi = \otimes_{k=1}^{\infty} h_{n_k}(x)$, there exists an M_0 independent of k such that $((a_k q_k + b_k p_k)^2 h_{n_k}, h_{n_k}) \leqslant M_0(a_k^2 + b_k^2)$. Thus

$$\sum_{k=1}^{\infty} ((a_k q_k + b_k q_k)^2 h_{n_k}, h_{n_k}) < \infty$$

which by Corollary 2.5 shows that $\sum_{k=1}^{\infty} a_k q_k + b_k p_k$ is self-adjoint on $H(\chi)$. The proof of Theorem 2.6 shows that χ is a strong convergence vector so by Theorem 2.2 $\sum_{k=1}^{\infty} a_k q_k + b_k p_k$ is essentially self-adjoint on D_0, the finite span of $\{\varphi = \otimes_{k=1}^{\infty} \varphi_k ; \varphi_k \in \mathscr{S}(R), \varphi_k = h_{n_k}$ for $k > M$, M arbitrary$\}$. If

$$(\{a_k^n\}_{k=1}^{\infty}, \{b_k^n\}_{k=1}^{\infty}) \xrightarrow{l_2 \times l_2} (\{a_k\}_{k=1}^{\infty}, \{b_k\}^{\infty})$$

and $\psi \in D_0$, then ψ has the form $\psi = \psi_M \otimes \otimes_{k=M+1}^{\infty} h_{n_k}(x)$ and

$$\begin{aligned}
&\left\| \sum_{k=1}^{\infty} (a_k^n q_k + b_k^n p_k) \psi - \sum_{k=1}^{\infty} (a_k q_k + b_k p_k) \psi \right\|^2 \\
&\quad \leqslant 2 \left\| \sum_{k=1}^{M} ((a_k^n - a_k) q_k + (b_k^n - b_k) p_k) \psi_M \right\|^2 \\
&\qquad + 2 \sum_{k=M+1}^{\infty} (((a_k^n - a_k) q_k + (b_k^n - b_n) p_k)^2 h_{n_k}, h_{n_k}) \| \psi_M \|^2 \\
&\quad \leqslant 2 \left\| \sum_{k-1}^{M} ((a_k^n - a_k) q_k + (b_n^n - b_k) p_k) \psi_M \right\|^2 \\
&\qquad + 4M_0 \| \psi_M \|^2 \sum_{k=M+1}^{\infty} (a_k^n - a_k)^2 + (b_k^n - b_k)^2.
\end{aligned}$$

The first term goes to zero as $n \to \infty$ since $a_k^n \to a_k$, $b_k^n \to b_k$ for each k, the second term because

$$(\{a_k^n\}_{k=1}^{\infty}, \{b_k^n\}_{k=1}^{\infty}) \xrightarrow{l_2 \times l_2} (\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty}).$$

5. The Free Field in Representations with Polynomially Bounded Occupation Numbers

If we quantize in a space cube of volume V, the formal expression for the time dependent free field of mass m is

$$\varphi(\mathbf{x}, t) = (1/\sqrt{V}) \sum_{\mathbf{k}} (1/\sqrt{2\omega_{\mathbf{k}}})\{\cos(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)\, q_{\mathbf{k}} - \sin(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)\, p_{\mathbf{k}}\},$$

where $\omega_{\mathbf{k}} = (\mathbf{k} \cdot \mathbf{k} + m^2)^{1/2}$, $\mathbf{k} = (k_1, k_2, k_3)$ runs over all 3-tuples such that $\cos(\mathbf{k} \cdot \mathbf{x})$ and $\sin(\mathbf{k} \cdot \mathbf{x})$ are periodic in B. If we use the $q_{\mathbf{k}}$ and $p_{\mathbf{k}}$ of the Fock representation, then it is well known that $\varphi(\cdot, t)$ is a well-defined operator valued distribution satisfying the Wightman axioms (for the discrete case). The following theorem says that if we choose for $\{q_{\mathbf{k}}, p_{\mathbf{k}}\}$ any representation with polynomially bounded occupation numbers, then $\varphi(\cdot, t)$ will be a well-defined time-dependent operator valued distribution with many of the same nice properties of the free field in the Fock representation. The test function space which we use is $C_p^{\infty}(B)$ the infinitely differentiable, real-valued, periodic functions on B. Because we have quantized in a box, the symmetry group is T^3 (the three-dimensional torus) which acts on $f(\mathbf{x}) \in C_p^{\infty}(B)$ by translating the argument modulo B. The representations with polynomially-bounded occupation number are just the natural p's and q's acting on the separable subspaces of $\otimes_{\mathbf{k}} L^2(R)$ generated by c_0-vectors of the form $\chi = \otimes_{\mathbf{k}} h_{n_{\mathbf{k}}}(x)$, $n_{\mathbf{k}}$ a nonnegative integer satisfying $|n_{\mathbf{k}}| \leqslant C_0(1 + |\mathbf{k}|)^r$ for some constant, C_0, and positive integer r. We remark that indexing the p's and q's by 3-tuples instead of positive integers makes no difference in the theorems in the previous sections.

Theorem 5.1. *Let $\{p_{\mathbf{k}}, q_{\mathbf{k}}\}_{\mathbf{k}}$ be a representation of the* CCR *with polynomially-bounded occupation numbers on a Hilbert space X. Then there exists a domain $D \subset X$ such that $\varphi(\cdot, t)$ is a time-dependent operator valued distribution from $C_p^{\infty}(B)$ to the self-adjoint operators on X satisfying:*

(1) *For all $f \in C_p^{\infty}(B)$ and $t \in R$, $\varphi(f, t)$ is essentially self-adjoint on D. If $\psi \in D$, $\varphi(f, t)\,\psi$ is an infinitely differentiable vector-valued function which satisfies*

$$(d^2/dt^2)\, \varphi(f, t)\, \psi = \varphi((\Delta - m^2) f, t)\, \psi$$

(2) *Let* $\pi(f, t) = (d/dt)\,\varphi(f, t)$. *Then for all* $f \in C_p^\infty(B)$ *and* $t \in R$, $\pi(f, t)$ *is essentially self-adjoint on* D. *Further, if* $f, g \in C_p^\infty(B)$, *then* $\varphi(f, t)\, D \subset D$, $\pi(g, t)\, D \subset D$,

$$[\varphi(f, t), \pi(g, t)] = i \int_B f(x)\, g(x)\, dx \qquad \text{on} \quad D$$

and the family $\{\varphi(f, t), \pi(g, t);\, f, g \in C_p^\infty(B)\}$ *is irreducible.*

(3) *If* $f_n \xrightarrow{C_p^\infty(B)} f$, *then* $\varphi(f_n, t)\,\psi \to \varphi(f, t)\,\psi$ *and*

$$\pi(f_n, t)\,\psi \to \pi(f, t)\,\psi \qquad \textit{for any} \quad \psi \in D.$$

(4) *There exists on* X *a strongly continuous unitary representation,* $U(a_1, a_2, a_3)\, V(t)$, *of* $R^3 \times R$ *such that* $U(a_1, a_2, a_3)\, D \subset D$, $V(t)\, D \subset D$, *and*

$$U(a_1, a_2, a_3)\,\varphi(f(\mathbf{x}), t)\,U(a_1, a_2, a_3)^{-1} = \varphi(f(\mathbf{x} + \mathbf{a}), t)$$
$$V(t)\,\varphi(f(\mathbf{x}), 0)\, V(t)^{-1} = \varphi(f(\mathbf{x}), t)$$

for all $f \in C_p^\infty(B)$ *and* $t \in R$. (By $\mathbf{x} + \mathbf{a}$ *we mean addition modulo* B.)

Proof. We sketch the proof. We may take X to be $H(\chi)$ where $\chi = \otimes_{\mathbf{k}} h_{n_{\mathbf{k}}}$, $|\, n_{\mathbf{k}} \,| \leqslant C_0(1 + |\, \mathbf{k} \,|)^r$. Let $S_3(\bar{S}_3)$ be the set of real-valued (complex-valued) rapidly decreasing sequences $\{a_{\mathbf{k}}\}_{\mathbf{k}}$, and let $\mathscr{S}(R)$ denote Schwartz space. We define

$$D_{00} = \Big\{\psi = \bigotimes_{\mathbf{k}} \psi_{\mathbf{k}}\,;\ \psi_{\mathbf{k}} \text{ a Hermite function, } \psi_{\mathbf{k}} = h_{n_{\mathbf{k}}} \text{ for } |\,\mathbf{k}\,| \geqslant M, M \text{ arbitrary}\Big\}.$$

$$D_0 = \text{finite span}\, \Big\{\psi = \bigotimes_{\mathbf{k}} \psi_{\mathbf{k}}\,;\ \psi_{\mathbf{k}} \in \mathscr{S}(R),\ \psi_{\mathbf{k}} = h_{n_{\mathbf{k}}} \text{ for } |\,\mathbf{k}\,| \geqslant M, M \text{ arbitrary}\Big\}.$$

$$D = \Big\{\sum_{\mathbf{m}} c_{\mathbf{m}}\psi^{\mathbf{m}};\ \{c_{\mathbf{m}}\} \in \bar{S}_3\,,\ \psi^{\mathbf{m}} = \bigotimes_{\mathbf{k}} h_{Q(\mathbf{m},\mathbf{k})} \in D_{00}\,, \text{ and } Q(\mathbf{m}, \mathbf{k}) \leqslant C(|\,\mathbf{k}\,| + 1)^{r_1}(|\,\mathbf{m}\,| + 1)^{r_2} \text{ for some positive integers } r_1, r_2 \text{ and constant } C\Big\}.$$

It is not hard to show that $D_{00} \subset D_0 \subset D \subset H(\chi)$.

Suppose $\{(a_{\mathbf{k}}, b_{\mathbf{k}})\}_{\mathbf{k}} \in S_3 \times S_3$, then a short computation (using the fact that $n_{\mathbf{k}} < C_0(1 + |\,\mathbf{k}\,|)^r$) shows that $\chi = \otimes_{\mathbf{k}} h_{n_{\mathbf{k}}}$ is a strong convergence vector for $\{a_{\mathbf{k}} q_{\mathbf{k}} + b_{\mathbf{k}} p_{\mathbf{k}}\}$ on $H(\chi)$. Therefore (by Theorem

2.2) $\sum_{\mathbf{k}} a_{\mathbf{k}} q_{\mathbf{k}} + b_{\mathbf{k}} p_{\mathbf{k}}$ is essentially self-adjoint on D_0, since $(a_{\mathbf{k}} q_{\mathbf{k}} + b_{\mathbf{k}} p_{\mathbf{k}})$ is essentially self-adjoint on $\mathscr{S}(R)$. Now, for $f \in C_p^{\infty}(B)$ define

$$a_{\mathbf{k}}(t) = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \int_B f(\mathbf{x}) \cos(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)\, d\mathbf{x},$$

$$b_{\mathbf{k}}(t) = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \int_B f(\mathbf{x}) \cos(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)\, d\mathbf{x},$$

and set $\varphi(f, t) = \sum_{\mathbf{k}} a_{\mathbf{k}}(t)\, q_{\mathbf{k}} + b_{\mathbf{k}}(t)\, p_{\mathbf{k}}$. Then because of the smoothness of $f(\mathbf{x})$, $\{(a_{\mathbf{k}}(t), b_{\mathbf{k}}(t))\}_{\mathbf{k}} \in S_3 \times S_3$ which implies that, for each t, $\varphi(f, t)$ is essentially self-adjoint on D_0. Now suppose $\psi = \sum_{\mathbf{m}} c_{\mathbf{m}} \psi^{\mathbf{m}} \in D$. Then $\sum_{|\mathbf{m}| \leqslant M} c_{\mathbf{m}} \psi^{\mathbf{m}} \in D_0$ for all M and $\sum_{|\mathbf{m}| \leqslant M} c_{\mathbf{m}} \psi^{\mathbf{m}} \to \sum_{\mathbf{m}} c_{\mathbf{m}} \psi^{\mathbf{m}}$. Since $\psi^{\mathbf{m}} = \bigotimes_{\mathbf{k}} h_{Q(\mathbf{m},\mathbf{k})}$ and

$$Q(\mathbf{m}, \mathbf{k}) \leqslant C(1 + |\, \mathbf{m} \,|)^{r_1} (1 + |\, \mathbf{k} \,|)^{r_2},$$

$$\begin{aligned}
\|(a_{\mathbf{k}}(t)\, q_{\mathbf{k}} + b_{\mathbf{k}}(t)\, p_{\mathbf{k}})\, \psi^{\mathbf{m}} \| &= \|(a_{\mathbf{k}}(t)\, q_{\mathbf{k}} + b_{\mathbf{k}}(t)\, p_{\mathbf{k}})\, h_{Q(\mathbf{m},\mathbf{k})} \| \\
&= |\, a_{\mathbf{k}}(t)| \left\| \left(\frac{Q(\mathbf{m}, \mathbf{k}) + 1}{2}\right)^{1/2} h_{Q(\mathbf{m},\mathbf{k})+1} + (\tfrac{1}{2} Q(\mathbf{m}, \mathbf{k}))^{1/2}\, h_{Q(\mathbf{m},\mathbf{k})-1} \right\| \\
&\quad + |\, b_{\mathbf{k}}(t)|\, \|(\tfrac{1}{2} Q(\mathbf{m}, \mathbf{k}))^{1/2}\, h_{Q(\mathbf{m},\mathbf{k})-1} - (\tfrac{1}{2} Q(\mathbf{m}, \mathbf{k}) + \tfrac{1}{2})^{1/2}\, h_{Q(\mathbf{m},\mathbf{k})+1} \| \\
&\leqslant (|\, a_{\mathbf{k}}(t)| + |\, b_{\mathbf{k}}(t)|)(M_0 C)(1 + |\, \mathbf{m} \,|)^{r_1/2} (1 + |\, \mathbf{k} \,|)^{r_2/2}.
\end{aligned} \tag{5.1}$$

It follows easily that $\varphi(f, t) \sum_{|\mathbf{m}| \leqslant M} c_{\mathbf{m}} \psi^{\mathbf{m}}$ converges as $M \to \infty$ which implies $D \subset D(\varphi(f, t))$ and $\varphi(f, t) | D$ is essentially self-adjoint. The reason for choosing D instead of the simpler domain D_0 is that $\varphi(f, t)\, D \subset D$ which may be seen as follows. Using the relations

$$x h_n(x) = \sqrt{\frac{n}{2}}\, h_{n-1}(x) + \sqrt{\frac{n+1}{2}}\, h_{n+1}(x)$$

$$\frac{d}{dx} h_n(x) = \sqrt{\frac{n}{2}}\, h_{n-1}(x) - \sqrt{\frac{n+1}{2}}\, h_{n+1}(x)$$

we can write

$$\varphi(f, t) \sum_{\mathbf{m}} c_{\mathbf{m}} \psi^{\mathbf{m}} = \sum_{\mathbf{m},\mathbf{k}} c_{\mathbf{m}} (a_{\mathbf{k}}(t)\, q_{\mathbf{k}} + b_{\mathbf{k}}(t)\, p_{\mathbf{k}}) \bigotimes_{\mathbf{l}} h_{Q(\mathbf{m}\ \mathbf{l})}$$

$$= \sum_{j=1}^{4} \sum_{\mathbf{m},\mathbf{k}} d^{j}_{\mathbf{m},\mathbf{k}} \psi_j^{\mathbf{m},\mathbf{k}}$$

where

$$|\, d^{j}_{\mathbf{m},\mathbf{k}} \,| \leqslant 2 \,|\, c_{\mathbf{m}} \,|\, (|\, a_{\mathbf{k}}(t)| + |\, b_{\mathbf{k}}(t)|)\, Q(\mathbf{m}, \mathbf{k})^{1/2} \tag{5.2}$$

and $\psi_j^{\mathbf{m},\mathbf{k}} = \bigotimes_{\mathbf{l}} h_{Q_j(\mathbf{m},\mathbf{k},\mathbf{l})}$ with $Q(\mathbf{m}, \mathbf{k}, \mathbf{l}) \leqslant Q(\mathbf{m}, \mathbf{l}) + 1$. The double sum $\sum_{\mathbf{m},\mathbf{k}} d^j_{\mathbf{m},\mathbf{k}} \psi_j^{\mathbf{m},\mathbf{k}}$ can be written as a single sum $\sum_{\mathbf{p}} e_{\mathbf{p}}{}^j \psi_j{}^{\mathbf{p}}$ so that if $\mathbf{p}_0$ corresponds to $(\mathbf{m}_0, \mathbf{k}_0)$,

$$c_1(|\mathbf{m}_0|^2 + |\mathbf{k}_0|^2) \leqslant |\mathbf{p}_0| \leqslant c_2(|\mathbf{m}_0|^2 + |\mathbf{k}_0|^2).$$

Thus $\{e_{\mathbf{p}}{}^j\}_{\mathbf{p}} \in \bar{S}_3$, since $\{d^j_{\mathbf{k},l}\} \in \bar{S}_3 \times \bar{S}_3$. Furthermore, if $\psi_j^{\mathbf{p}_0} = \bigotimes_{\mathbf{l}} h_{Q_j(\mathbf{p}_0, l)}$, then

$$\begin{aligned} Q_j(\mathbf{p}_0, l) = Q_j(\mathbf{m}_0, \mathbf{k}_0, l) &\leqslant C(1 + |\mathbf{m}_0|)^{r_1} (1 + |l|)^{r_2} + 1 \\ &\leqslant C'(1 + |\mathbf{p}_0|)^{r_1/2} (1 + |l|)^{r_2}. \end{aligned}$$

Thus $\varphi(f, t) \sum_{\mathbf{m}} c_{\mathbf{m}} \psi^{\mathbf{m}} = \sum_{j=1}^4 \sum_l e_l \psi_j{}^l \in D$.

Suppose

$$\{(a_{\mathbf{k}}{}^n, b_{\mathbf{k}}{}^n)\}_{\mathbf{k}} \xrightarrow[n\to\infty]{S_3\times S_3} \{(a_{\mathbf{k}}, b_{\mathbf{k}})\}_{\mathbf{k}},$$

then if $\psi = \sum_{\mathbf{m}} c_{\mathbf{m}} \psi^{\mathbf{m}} \in D$,

$$\begin{aligned} &\Big\| \sum_{\mathbf{k}} (a_{\mathbf{k}}{}^n q_{\mathbf{k}} + b_{\mathbf{k}}{}^n p_{\mathbf{k}}) \psi - \sum_{\mathbf{k}} (a_{\mathbf{k}} q_{\mathbf{k}} + b_{\mathbf{k}} p_{\mathbf{k}}) \psi \Big\| \\ &\qquad \leqslant \sum_{\mathbf{k},\mathbf{m}} |c_{\mathbf{m}}| (|a_{\mathbf{k}}{}^n - a_{\mathbf{k}}| + |b_{\mathbf{k}}{}^n - b_{\mathbf{k}}|)(\| q_{\mathbf{k}} \psi^{\mathbf{m}} \| + \| p_{\mathbf{k}} \psi^{\mathbf{m}} \|) \\ &\qquad \leqslant \sum_{\mathbf{k},\mathbf{m}} |c_{\mathbf{m}}| (|a_{\mathbf{k}}{}^n - a_{\mathbf{k}}| + |b_{\mathbf{k}}{}^n - b_{\mathbf{k}}|)(4)(\tfrac{1}{2} Q(\mathbf{k}, \mathbf{m}) + \tfrac{1}{2})^{1/2} \end{aligned}$$

and this goes to zero as $n \to \infty$ since $Q(\mathbf{k}, \mathbf{m})$ is polynomially bounded and $\{c_{\mathbf{m}}\} \in \bar{S}_3$. Now, if $f_n \xrightarrow{C_p{}^\infty(B)} f$, then integration by parts and a simple estimate show that

$$\{(a_{\mathbf{k}}{}^n(t), b_{\mathbf{k}}{}^n(t)\}_{\mathbf{k}} \xrightarrow[n\to\infty]{S_3\times S_3} \{a_{\mathbf{k}}(t), b_{\mathbf{k}}(t)\}_{\mathbf{k}}$$

so $\varphi(f_n, t)$ converges strongly to $\varphi(f, t)$ on D.

To prove the differentiability of $\varphi(f, t) \psi$ we observe that

$$\begin{aligned} &\Big| \mathbf{k}^\alpha \frac{a_{\mathbf{k}}(t+s) - a_{\mathbf{k}}(t)}{s} - a_{\mathbf{k}}'(t) \Big| \\ &\quad \leqslant \Big(\frac{V}{2\omega_{\mathbf{k}}}\Big)^{1/2} \Big(\sup_B \Big| \frac{\partial^\alpha}{\partial x^\alpha} f \Big|\Big) \\ &\qquad \times \sup_B \Big| \frac{\cos(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}}(t+s)) - \cos(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)}{s} \\ &\qquad - \omega_{\mathbf{k}} \sin(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t) \Big| \\ &\quad \leqslant \Big(\frac{V}{2\omega_{\mathbf{k}}}\Big)^{1/2} \Big(\sup_B \Big| \frac{\partial^\alpha}{\partial x^\alpha} f \Big|\Big) (\omega_{\mathbf{k}}{}^2 |s - t|) \end{aligned}$$

Since α was arbitrary, $\{[a_{\mathbf{k}}(t+s) - a_{\mathbf{k}}(t)]/s\}_{\mathbf{k}} \xrightarrow{S_3} \{a_{\mathbf{k}}'(t)\}_{\mathbf{k}}$ and similarly $\{[b_{\mathbf{k}}(t+s) - b_{\mathbf{k}}(t)]/s\}_{\mathbf{k}} \xrightarrow{S_3} \{b_{\mathbf{k}}'(t)\}_{\mathbf{k}}$. Thus, $\{[\varphi(f,\ t+s) - \varphi(t)]/s\}\,\psi$ converges to $\sum_{\mathbf{k}} (a_{\mathbf{k}}'(t)\, q_{\mathbf{k}} + b_{\mathbf{k}}'(t)\, p_{\mathbf{k}})\,\psi$ as $s \to 0$. Repeating the same proof shows that $\varphi(f, t)\,\psi$ is infinitely differentiable. A simple integration by parts shows that

$$\begin{aligned}
\frac{d^2}{dt^2}\,\varphi(f, t)\,\psi &= \sum_{\mathbf{k}} \frac{-1}{\sqrt{2V\omega_{\mathbf{k}}}} \Big\{\Big(\int f(\mathbf{x})\, \omega_{\mathbf{k}}^2 \cos(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)\, d\mathbf{x}\Big)\, q_{\mathbf{k}} \\
&\quad + \Big(\int f(\mathbf{x})\, \omega_{\mathbf{k}}^2 \sin(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)\, d\mathbf{x}\Big)\, p_{\mathbf{k}}\Big\} \\
&= \sum_{\mathbf{k}} \Big(\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\Big)\Big\{\Big(\int ((\Delta - m^2) f(x)) \cos(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)\, d\mathbf{x}\Big)\, q_{\mathbf{k}} \\
&\quad + \Big(\int ((\Delta - m^2) f(\mathbf{x})) \sin(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)\, d\mathbf{x}\Big)\, p_{\mathbf{k}}\Big\} \\
&= \varphi((\Delta - m^2) f, t).
\end{aligned}$$

We now define $\pi(f, t) = d/dt\, \varphi(f, t)$. The proofs of the properties of $\pi(f, t)$ are similar to the proofs for $\varphi(f, t)$ and the fact that $[\varphi(f, t), \pi(g, t)] = i \int f(\mathbf{x})\, g(\mathbf{x})\, d\mathbf{x}$ is a straightforward calculation. By taking linear combinations of $\varphi(f, t)$ and $\pi(g, t)$ with appropriate test functions we can recover all the operators $\{q_{\mathbf{k}}, p_{\mathbf{k}}\}$. Since this set is irreducible [8], so is the family $\{\varphi(f, t), \pi(g, t);\ f, g \in C_p^{\infty}(B,)\}$.

Let $\{\lambda_{\mathbf{k}}\}_{\mathbf{k}}$ be any sequence of real numbers. Then $\chi = \bigotimes_{\mathbf{k}} h_{n_{\mathbf{k}}}(x)$ is a strong convergence vector for $\{\lambda_{\mathbf{k}}(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1))\}_{\mathbf{k}}$ in $H(\chi)$. Since $\lambda_{\mathbf{k}}(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1))$ is essentially self-adjoint on $\mathscr{S}(R)$, Theorem 2.2 shows that $\sum_{\mathbf{k}} \lambda_{\mathbf{k}}(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1))$ is essentially self-adjoint on D_0. If $\psi = \sum_{\mathbf{m}} c_{\mathbf{m}} \psi^{\mathbf{m}} \in D$ then

$$\| \lambda_{\mathbf{k}}(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1)\, \psi^{\mathbf{m}} \| \leqslant 2\, |\lambda_{\mathbf{k}}|\, (Q(\mathbf{k}, \mathbf{m}) + C_0(1 + |\mathbf{k}|)^r)$$

from which it follows that $D \subset D\{\sum_{\mathbf{k}} \lambda_{\mathbf{k}}(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1))\}$ if $|\lambda_{\mathbf{k}}|$ is polynomially bounded. Theorem 2.6 shows that the group generated by $\sum_{\mathbf{k}} \lambda_{\mathbf{k}}(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1))$ is

$$U_\lambda(t) = \bigotimes_{\mathbf{k}} \exp(i\lambda_{\mathbf{k}} t(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1))).$$

Since $U_\lambda(t)\,\psi = \sum_{\mathbf{m}} c_{\mathbf{m}} U_\lambda(t)\,\psi^{\mathbf{m}} = \sum_{\mathbf{m}} c_{\mathbf{m}} z_{\mathbf{m}} \psi^{\mathbf{m}}$ where $|z_{\mathbf{m}}| = 1$, we have $U_\lambda(t)\, D \subset D$.

We now define $H_0 = \sum_{\mathbf{k}} (\omega_{\mathbf{k}}/2)(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1))$

$$P_i = \sum_{\mathbf{k}} k_i(p_{\mathbf{k}}^2 + q_{\mathbf{k}}^2 - (2n_{\mathbf{k}} + 1)), \qquad i = 1, 2, 3, \quad \mathbf{k} = (k_1, k_2, k_3).$$

H_0 and $\{P_i\}$ are essentially self-adjoint on D and D is invariant under $U(a_1, a_2, a_3) = \exp(i \sum_{i=1}^{3} a_i \cdot P_i)$ and $V(t) = \exp(-itH_0)$. A short computation with Hermite functions shows that on $L^2(R)$

$$\exp\left(i\lambda\left(\frac{-d^2}{dx^2} + x^2\right)\right)\left(ax + \frac{b}{i}\frac{d}{dx}\right)\exp\left(-i\lambda\left(\frac{-d^2}{dx^2} + x^2\right)\right)$$
$$= (a\cos\lambda - b\sin\lambda)\, x + (a\sin\lambda + b\cos\lambda)\frac{1}{i}\frac{d}{dx}.$$

The properties of $\varphi(f, t)$ stated in 4) follow immediately. This completes the proof.

We remark that the Fock representation differs from the other representations with polynomially-bounded occupation numbers in two ways. First of all, in each representation there is a vector (namely $\chi = \otimes_k h_{n_k}(x)$, the vector which generates the space) which is invariant under the corresponding $U(a_1, a_2, a_2)$ and $V(t)$ and is thus a candidate to be called the "vacuum." But, only in the Fock representation will it be annihilated by all the operators $\{q_k + ip_k\}_k$ and furthermore the vacuum expectation values computed with $\varphi(f, t)$ and $\pi(g, t)$ using this vacuum, equal the vacuum expectation values of the "free" field only in the Fock representation. Secondly, it is not hard to see that the generator of $V(t)$ will be semi-bounded only in Fock representation (it follows immediately from Theorem 3.3). Therefore, $U(a_1, a_2, a_3)\, V(t)$ will satisfy the spectral condition only in the Fock representation. We note that the proof of Theorem 5.1 implies the result of Klauder and McKenna [3] that the maps $\mathscr{S}(R^3) \to e^{i\varphi(f)}$, $\mathscr{S}(R^3) \to e^{i\pi(f)}$ make sense and are continuous in the representations with polynomial-bounded occupation numbers.

6. Hamiltonians

The theorems of Sections 2 and 3 apply directly to any Hamiltonian in diagonal form, that is, of the form $\sum_{k=1}^{\infty} A_k$ where A_k is a function only of p_k and q_k. Unfortunately, interaction Hamiltonians of this form have only pedagogical interest; all physically interesting Hamiltonians have cross terms. Nevertheless, we can treat some of these nondiagonal examples if the coefficients of the off-diagonal terms are small enough. For example, consider the formal operator

$$\sum_{k=1}^{\infty} \frac{\omega_k}{2}(p_k^2 + q_k^2) + \sum_{k,l,m,n} d_{k,l,m,n} q_k q_l q_m q_n.$$

We write it in the form

$$\sum_{k=1}^{\infty} \frac{\omega_k}{2} (p_k^2 + q_k^2 + d_{kkkk} q_k^4) + \sum_{k,l,m,n}{}' d_{k,l,m,n} q_k q_l q_m q_n$$

(the prime means there are no terms where $k = l = m = n$). Now suppose the d_{kkkk} are nonnegative, but otherwise arbitrary. Then $(d^2/dx^2) + x^2 + d_{kkkk} x^4$ is essentially self-adjoint on $\mathscr{S}(R) \subset L^2(R)$ and has point spectrum, τ_k, of multiplicity one as lowest point in its spectrum; let χ_k be the corresponding eigenfunction. If we choose for $\{q_k, p_k\}$ the representation of the CCR on the infinite tensor product space generated by $\chi = \otimes_{k=1}^{\infty} \chi_k$, then

$$\sum_{k=1}^{\infty} \frac{\omega_k}{2} (p_k^2 + q_k^2 + d_{kkkk} q_k^4 - \tau_k)$$

will be essentially self-adjoint on the domain $D_\chi \subsetneq H(\chi)$ (Theorem 2.2). If the off-diagonal d_{klmn} are sufficiently small then $\Sigma' d_{klmn} q_k q_l q_m q_n$ can be estimated in terms of the diagonal part (in the sense of Kato). The whole operator (with the subtractions $\{\tau_k\}$ will then be essentially self-adjoint on D_χ. Further analysis of the spectrum of the whole operator can be carried out be treating it as an analytic perturbation of the diagonal part (for more details see [7]). The smallness conditions on the off-diagonal coefficients are too strong to include the physical case where

$$d_{k,l,m,n} \sim \frac{1}{\sqrt{\omega_k \omega_l \omega_m \omega_n}} .$$

Nevertheless, our Hamiltonian has an infinite number of connected modes which, in some sense, gives a better approximation to the "physical" Hamiltonian than cutting off the high modes altogether.

Acknowledgment

Some of this work forms part of the author's doctoral dissertation at Stanford University. He is grateful for the encouragement and insight of his adviser, Professor Ralph Phillips. Several helpful suggestions were made by Professor Edward Nelson of Princeton University.

Appendix

The Infinite Tensor Product of Hilbert Spaces

In this Appendix we briefly describe the construction of the infinite tensor product of Hilbert spaces, due to von Neumann; [6].

Let $\{H_\alpha\}_{\alpha \in I}$ be a family of Hilbert spaces, I a not necessarily countable index set. Let $z_\alpha : I \to \mathbb{C}$.

DEFINITION. $\prod_{\alpha \in I} z_\alpha$ is said to *converge to a limit* ξ if given $\delta > 0$, there exists a finite set $I_\delta \subseteq I$ such that for all finite sets J, $I_\delta \subseteq J \subseteq I$, we have

$$\left| \prod_{\alpha \in J} z_\alpha - \xi \right| < \delta.$$

DEFINITION. $\prod_{\alpha \in I} z_\alpha$ is said to be quasi-convergent if $\prod_{\alpha \in I} | z_\alpha |$ is convergent. If $\prod_{\alpha \in I} z_\alpha$ is quasi-convergent its value is defined as:

$$\begin{cases} \prod_{\alpha \in I} z_\alpha & \text{if } \prod_{\alpha \in I} z_\alpha \text{ is convergent} \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION. Suppose $f_\alpha \in H_\alpha$ for each $\alpha \in I$ and

$$\sum_{\alpha \in I} | \| f_\alpha \|_{H_\alpha} - 1 | < \infty,$$

then the symbol $\bigotimes_{\alpha \in I} f_\alpha$ is called a c_0-vector.

We note that if $\bigotimes f_\alpha$ is a c_0-vector then $\prod \| f_\alpha \|$ converges and is not zero unless some f_α is the zero vector (we have dropped the $\alpha \in I$ in products and tensor products). As a scalar product on these c_0-vectors we take

$$(\bigotimes f_\alpha, \bigotimes g_\alpha) = \prod (f_\alpha, g_\alpha)_{H_\alpha}.$$

That this scalar product makes sense follows from

LEMMA A.1.1. *If* $\prod \| f_\alpha \|$ *converges, so does* $\prod \| f_\alpha \|^2$.

LEMMA A.1.2. *If* $\prod \| f_\alpha \|$ *and* $\prod \| g_\alpha \|$ *converge then* $\prod (f_\alpha, g_\alpha)$ *is quasi-convergent.*

We now extend this scalar product in the natural way to finite linear combinations of c_0-vectors. The proof that it is positive semi-definite on such finite linear combinations is nontrivial and uses the following notion of equivalence.

DEFINITION. Two c_0-vectors $\bigotimes f_\alpha$ and $\bigotimes g_\alpha$ are said to be equivalent (written $\sim$) if $\sum |(f_\alpha, g_\alpha) - 1| < \infty$.

LEMMA A.1.3: $\sim$ *is an equivalence relation.*

LEMMA A.1.4. *If* $\otimes f_\alpha$ *and* $\otimes g_\alpha$ *are two* c_0*-vectors which belong to different equivalence classes* $(\otimes f_\alpha, \otimes g_\alpha) = 0$. *If they belong to the same equivalence class then* $(\otimes f_\alpha, \otimes g_\alpha) = 0$ *if and only if* $(f_\alpha, g_\alpha)_{H_\alpha} = 0$ *for some* $\alpha \in I$.

LEMMA A.1.5. $(\cdot, \cdot)$ *is positive definite on finite linear combinations of* c_0*-vectors.*

DEFINITION. The infinite tensor product of the Hilbert spaces H_α (written $H = \otimes H_\alpha$) is the completion of the set of finite linear combinations of c_0-vectors under the scalar product $(\cdot, \cdot)$.

Let E be the set of equivalence classes of c_0-vectors, then from Lemma A.1.4 it is clear that H can be written as a direct sum $H = \Sigma \otimes_{e \in E} H_e$ where H_e is the closure of the finite linear combinations of c_0-vectors in the equivalence class e. Actually the following lemma shows that we may think of H_e as being "generated" by a single c_0-vector in e.

LEMMA A.1.6. *Let e be an equivalence class of* c_0*-vectors and* $\chi = \otimes \chi_\alpha \in e$. *Then the finite span of*

$$\{\psi = \otimes \psi_\alpha, \psi_\alpha = \chi_\alpha \text{ except for a finite number of } \alpha \in I\}$$

is dense in H_e.

We will therefore write $H(\chi)$ instead of H_e. That is $H(\chi)$ is the subspace of H generated by the c_0-vector $\chi = \otimes \chi_\alpha$. We have of course $H = \sum \otimes_\chi H(\chi)$ where we allow in the sum only one χ from each equivalence class. We will refer to the subspaces $H(\chi)$ as infinite tensor product spaces and to H as the infinite tensor product of the Hilbert spaces H_α. We have also:

LEMMA A.1.7. *In each equivalence class there is a* c_0*-vector*

$$\chi = \otimes \chi_\alpha \quad \text{with} \quad \| \chi_\alpha \|_{H_\alpha} = 1 \quad \text{for all} \quad \alpha \in I.$$

LEMMA A.1.8. *If I is countable and each* H_α *is separable, then each of the infinite tensor product spaces* $H(\chi)$ *is separable.*

Finally we have the notion of weak equivalence.

DEFINITION. Two c_0-vectors $\otimes f_\alpha$ and $\otimes g_\alpha$ are said to be weakly equivalent (written $\approx$) if $\sum \big| |(f_\alpha, g_\alpha)| - 1 \big| < \infty$.

LEMMA A.1.9: $\approx$ *is an equivalence relation.*

We note that weak equivalence is weaker than equivalence.

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